

# Design of Symmetric Bi-orthogonal Double Density Wavelet Filter Banks

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**Abstract**—We look at the design of a class of oversampled filter banks and the resulting framelets. The oversampled property is achieved via an extra subband resulting in Double Density Filter Banks (DDFB's). We design a class of such filters with linear phase property.

We will look at a special class of framelets from a filter bank perspective, in that we will design double density filter banks (DDFB's) as shown in Figure 1. We define type 1 polyphase representation as

$$X(z) = \sum_{k=0}^1 z^{-k} X_k(z^2) \quad (1)$$

and type 2 polyphase representation as

$$X(z) = \sum_{k=0}^1 z^k X_k(z^2). \quad (2)$$

Given the following polyphase matrices:

$$\tilde{H}(z) = \begin{bmatrix} \tilde{H}_{00}(z) & \tilde{H}_{01}(z) \\ \tilde{H}_{10}(z) & \tilde{H}_{11}(z) \\ \tilde{H}_{20}(z) & \tilde{H}_{21}(z) \end{bmatrix}, H(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \\ H_{20}(z) & H_{21}(z) \end{bmatrix} \quad (3)$$

where  $\tilde{H}(z)$  is the type 1 analysis polyphase matrix, and  $H(z)$  is the type 2 synthesis polyphase matrix, we can write the perfect reconstruction conditions as

$$[H(z)]^T \tilde{H}(z) = I. \quad (4)$$

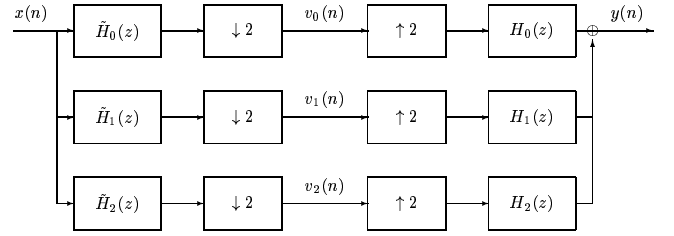


Fig. 1. The double density filter bank

## I. A POLYPHASE BASED DESIGN APPROACH

As with many problems in signal processing, polyphase constructions substantially simplify constraints. It is the same with double density filter banks. Let the analysis and synthesis wavelet filters are of the form

$$\begin{aligned} \tilde{H}_1(z) &= (1 - z)^{\tilde{K}_h} \tilde{A}_1(z), \\ \tilde{H}_2(z) &= (1 - z)^{\tilde{K}_h} \tilde{A}_2(z), \\ H_1(z) &= (1 - z^{-1})^{K_h} A_1(z), \\ H_2(z) &= (1 - z^{-1})^{K_h} A_2(z). \end{aligned} \quad (5)$$

Let the type 1 polyphase matrix of analysis wavelet filters and type 2 polyphase matrix of synthesis wavelet filters are given by

$$\tilde{H}_h(z) = \begin{bmatrix} \tilde{H}_{10}(z) & \tilde{H}_{11}(z) \\ \tilde{H}_{20}(z) & \tilde{H}_{21}(z) \end{bmatrix}$$

and

$$H_h(z) = \begin{bmatrix} H_{10}(z) & H_{11}(z) \\ H_{20}(z) & H_{21}(z) \end{bmatrix}.$$

Then from equation 4 it is easy to see that

$$H_h^T(z)\tilde{H}_h(z) = I - \begin{bmatrix} H_{00}(z) \\ H_{01}(z) \end{bmatrix} \begin{bmatrix} \tilde{H}_{00}(z) & \tilde{H}_{01}(z) \end{bmatrix}. \quad (6)$$

We can obtain a simplified expression for both  $\tilde{H}_h(z)$  and  $H_h(z)$ . Note that the type 1 polyphase vector of  $(1-z)S(z)$  can be written as  $\begin{bmatrix} 1 & -1 \\ -z & 1 \end{bmatrix} S_p(z)$  where  $S_p(z)$  is the type 1 polyphase vector of  $S(z)$ . Thus the type 1 polyphase vector  $\begin{bmatrix} \tilde{H}_{10}(z) \\ \tilde{H}_{11}(z) \end{bmatrix}$  can be written in the following form.

$$\begin{bmatrix} \tilde{H}_{10}(z) \\ \tilde{H}_{11}(z) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -z & 1 \end{bmatrix}^{\tilde{K}_h} \tilde{B}_1(z).$$

Similarly

$$\begin{bmatrix} \tilde{H}_{20}(z) \\ \tilde{H}_{21}(z) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -z & 1 \end{bmatrix}^{\tilde{K}_h} \tilde{B}_2(z).$$

Thus

$$\tilde{H}_h(z) = \tilde{A}(z) \begin{bmatrix} 1 & -z \\ -1 & 1 \end{bmatrix}^{\tilde{K}_h} \quad (7)$$

where  $\tilde{A}(z) = \begin{bmatrix} \tilde{B}_1^T(z) \\ \tilde{B}_2^T(z) \end{bmatrix}$ . Similarly

$$H_h(z) = A(z) \begin{bmatrix} 1 & -z^{-1} \\ -1 & 1 \end{bmatrix}^{\tilde{K}_h}. \quad (8)$$

Let

$$P(z) = \begin{bmatrix} 1 & 1 \\ z^{-1} & 1 \end{bmatrix}^{\tilde{K}_h} \begin{bmatrix} I - \begin{bmatrix} H_{00}(z) \\ H_{01}(z) \end{bmatrix} \begin{bmatrix} \tilde{H}_{00}(z) & \tilde{H}_{01}(z) \end{bmatrix} \\ \begin{bmatrix} 1 & z \\ 1 & 1 \end{bmatrix}^{\tilde{K}_h} \end{bmatrix} \quad (9)$$

and  $Q(z) = \frac{1}{(1-z^{-1})^{\tilde{K}_h}(1-z)^{\tilde{K}_h}} P(z)$ . Since

$$\begin{bmatrix} 1 & -z \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & z \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1-z & 0 \\ 0 & 1-z \end{bmatrix}, \quad (10)$$

$$\begin{bmatrix} 1 & 1 \\ z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -z^{-1} & 1 \end{bmatrix} = \begin{bmatrix} 1-z^{-1} & 0 \\ 0 & 1-z^{-1} \end{bmatrix}$$

we get

$$A^T(z)\tilde{A}(z) = Q(z). \quad (11)$$

Many factorization methods of Laurent polynomial matrices are crucially dependent on the determinant of the matrix. Fortunately, for a good subclass of double density filter banks, we have a simplified result for the determinant of  $Q(z)$ . We assume that the number of vanishing moments of analysis high pass filters and synthesis high pass filters are equal, i.e.  $K_h = \tilde{K}_h = M$ . It can be seen that the determinant of  $H_h^T(z)\tilde{H}_h(z)$  is given by  $D(z) = 1 - H_{00}(z)\tilde{H}_{00}(z) - H_{01}(z)\tilde{H}_{01}(z)$ . Since  $Q(z)$  is FIR and both  $\begin{bmatrix} 1 & -z \\ -1 & 1 \end{bmatrix}^M$  and  $\begin{bmatrix} 1 & -1 \\ -z^{-1} & 1 \end{bmatrix}^M$  are factors of  $H_h^T(z)\tilde{H}_h(z)$ , it must be that  $(1-z)^M(1-z^{-1})^M$  is a factor of  $D(z)$ . We have the following Lemma [2].

*Lemma 1:* Let the high pass filters has  $M$  number of vanishing moments each and the low pass filters,  $H_0$  and  $\tilde{H}_0$ , have the equal regularity,  $K$ , and given by [4], [5]

$$H_0(z)\tilde{H}_0(z) = 2 \left( \frac{z+2+z^{-1}}{4} \right)^K \sum_{n=0}^{M-1} \binom{K+n-1}{n} \left( \frac{-z+2-z^{-1}}{4} \right)^n. \quad (12)$$

Then the determinant of  $Q(z)$  is a real number when,

$$M = \left\lfloor \frac{K+M-1}{2} \right\rfloor. \quad (13)$$

## II. SYMMETRIC HIGH PASS FILTERS

We consider FIR-linear phase wavelet filters with the polyphase vector of the form  $[x(z) \quad z^k x^R(z)]$  where  $x^R(z)$  is the reverse of  $x(z)$ . Note, for  $x.y(z) = x(z)y(z)$ ,  $(x.y)^R(z) = x^R(z)y^R(z)$  and if the centers of polynomial index ranges of  $x(z)$  and  $y(z)$  are the same, for  $(x \pm y)(z) = x(z) \pm y(z)$ ,  $(x \pm y)^R(z) = x^R(z) \pm y^R(z)$ . We consider FIR-linear phase polyphase wavelet matrices of the form  $P \begin{bmatrix} x_0(z) & z^k x_0^R(z) \\ x_1(z) & -z^k x_1^R(z) \end{bmatrix}$  where  $P$  is a permutation matrix. Thus both wavelet filters are of same size and have the same symmetric axis and exactly one filter is symmetric and the other

is anti-symmetric. Also note that the analysis filters given by the polyphase matrix

$$P \begin{bmatrix} x_0(z) & z^k x_0^R(z) \\ x_1(z) & -z^k x_1^R(z) \end{bmatrix} \begin{bmatrix} 1 & -z \\ 1 & 1 \end{bmatrix}^K$$

are also FIR-linear phase symmetric and similarly for the synthesis wavelet filters. Thus in designing linear phase wavelet filters we consider design of linear phase  $\tilde{A}(z)$  and  $A(z)$  that are given in the equations 7 and 8.

We have the following factorization for FIR-invertible linear phase two band filter banks, which is motivated by the results reported in [1].

*Theorem 1:* Let  $A(z)$  be two band FIR-invertible linear phase polyphase matrix of the form

$$P \begin{bmatrix} x_0(z) & z^k x_0^R(z) \\ x_1(z) & -z^k x_1^R(z) \end{bmatrix}$$

where  $P$  is a 2x2 permutation matrix then it may be factorized as

$$P \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left\{ \prod_{i=1}^K \begin{bmatrix} a_i & z^{-1}b_i \\ b_i & z^{-1}a_i \end{bmatrix} \right\} \begin{bmatrix} a_0 & b_0 \\ b_0 & a_0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^k \end{bmatrix}. \quad (14)$$

*Proof:* We can see that

$$\begin{bmatrix} x_0(z) & z^k x_0^R(z) \\ x_1(z) & -z^k x_1^R(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_0(z) & y_1(z) \\ y_1^R(z) & y_0^R(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^k \end{bmatrix}. \quad (15)$$

We will show that  $\begin{bmatrix} y_0(z) & y_1(z) \\ y_1^R(z) & y_0^R(z) \end{bmatrix}$  can be written as

$$\left\{ \prod_{i=1}^K \begin{bmatrix} a_i & z^{-1}b_i \\ b_i & z^{-1}a_i \end{bmatrix} \right\} \begin{bmatrix} a_0 & b_0 \\ b_0 & a_0 \end{bmatrix} = X(z).$$

Let

$$X(z) = \sum_{i=0}^K x(i)z^{-i} = \sum_{i=0}^K \begin{bmatrix} y_0(i) & y_1(i) \\ y_1(K-i) & y_0(K-i) \end{bmatrix}.$$

We will find an invertible matrix  $\begin{bmatrix} c_K & -d_K \\ -d_K & c_K \end{bmatrix}$  such that  $\begin{bmatrix} c_K & -d_K \end{bmatrix} x(K) = 0$ . Our method succeeds only if  $c_K^2 \neq$

$d_K^2$ . Then we will get a degree reduction since

$$\begin{bmatrix} c_K & -d_K \\ -z d_K & z c_K \end{bmatrix} X(z) = \sum_{i=0}^{K-1} z^{-i} \begin{bmatrix} c_K y_0(i) - d_K y_1(K-i) & c_K y_1(i+1) - d_K y_0(K-1-i) \\ c_K y_1(K-i) - d_K y_0(i) & c_K y_0(K-1-i) - d_K y_1(i+1) \end{bmatrix}.$$

Let  $W(z)$  is the inverse of  $X(z)$  and w.l.o.g  $W(z) = \sum_{i=0}^K w(i)z^i$ . Then we get  $w(0)x(K) = 0$ . Since it must be that the rank of  $x(K)$  and  $w(0)$  is one, there exists  $\begin{bmatrix} c_K & -d_K \end{bmatrix}$  such that  $\begin{bmatrix} c_K & -d_K \end{bmatrix} x(K) = 0$ . ■

Not all the matrices  $Q(z) = A^T(z)\tilde{A}(z)$  can be factorized to result in linear phase wavelet filters. The following Theorem identifies  $Q(z)$  with a real number determinant that can be factorized to result in linear phase wavelet filters.

*Theorem 2:* Let

$$\tilde{A}(z) = P \begin{bmatrix} \tilde{x}_0(z) & z^{\tilde{k}} \tilde{x}_0^R(z) \\ \tilde{x}_1(z) & -z^{\tilde{k}} \tilde{x}_1^R(z) \end{bmatrix}$$

and

$$A(z) = P \begin{bmatrix} x_0(z) & z^k x_0^R(z) \\ x_1(z) & -z^k x_1^R(z) \end{bmatrix}.$$

Then  $Q(z)$  is of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & z^k \end{bmatrix} \begin{bmatrix} y_0(z) & y_1(z) \\ y_1^R(z) & y_0^R(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{\tilde{k}} \end{bmatrix}.$$

*Proof:* The result follows since  $\tilde{A}(z)$  can be written as

$$P \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{w}_0(z) & \tilde{w}_1(z) \\ \tilde{w}_1^R(z) & \tilde{w}_0^R(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{\tilde{k}} \end{bmatrix}$$

and  $A(z)$  can be written as

$$P \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} w_0(z) & w_1(z) \\ w_1^R(z) & w_0^R(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^k \end{bmatrix}. \quad \blacksquare$$

*Example 1:* Consider the (3,2) bi-orthogonal-like DDFB with  $H_0(z) = \frac{\sqrt{2}}{8}(1+z)^3$  and  $\tilde{H}_0(z) = \frac{1}{16\sqrt{2}}(1+z^{-1})^3(-3z+8-3z^{-1})$ . Now  $A^T(z)\tilde{A}(z)$  is given by

$$\frac{1}{128} \begin{bmatrix} 25 + 3z^{-1} & 5 + 15z \\ 15z^{-1} + 5 & 3z + 25 \end{bmatrix}$$

which can be factorized as

$$\frac{1}{128} \begin{bmatrix} 5 & z^{-1} \\ 1 & 5z^{-1} \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

Analysis filters	Synthesis filters
$30z^{-5} - 60z^{-4} + 48z^{-3} - 18z^{-2} - 18z^{-1} + 48 - 60z + 30z^2$	$\frac{1}{256}(z^{-2} - z^{-1} - 1 + z)$
$-20z^{-5} + 40z^{-4} - 8z^{-3} - 36z^{-2} + 36z^{-1} + 8 - 40z + 20z^2$	$\frac{1}{256}(z^{-2} - 3z^{-1} + 3 - z)$
$z^{-3} - 2z^{-2} + z^{-1} + 1 - 2z + z^2$	$\frac{1}{128}(4z^{-4} + 12z^{-3} - 16z^{-2} - 16z^{-1} + 12 + 4z)$
$-z^{-3} + 2z^{-2} - z^{-1} + 1 - 2z + 2z^2$	$\frac{1}{128}(-z^{-4} - 3z^{-3} + 14z^{-2} - 14z^{-1} + 3 + z)$
$z^{-1} - 1 - z + z^2$	$\frac{1}{256}(5z^{-6} + 15z^{-5} - 27z^{-4} + 7z^{-3} + 7z^{-2} - 27z^{-1} + 15 + 5z)$
$-z^{-1} + 3 - 3z + z^2$	$\frac{1}{256}(-5z^{-6} - 15z^{-5} + 33z^{-4} + 11z^{-3} - 11z^{-2} - 33z^{-1} + 15 + 5z)$

TABLE I

WAVLET FILTERS FOR THE (3,2) DDFB WITH  $H_0(z) = \frac{\sqrt{2}}{8}(1+z)^3$  AND

$$\tilde{H}_0(z) = \frac{1}{16\sqrt{2}}(1+z^{-1})^3(-3z+8-3z^{-1}).$$

Now there are three possibilities for wavelet filters:

$$\begin{aligned} \tilde{A}(z) &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & z^{-1} \\ 1 & 5z^{-1} \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \\ A(z) &= \frac{1}{256} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \end{aligned} \quad (16)$$

or

$$\begin{aligned} \tilde{A}(z) &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \\ A(z) &= \frac{1}{256} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ z^{-1} & 5z^{-1} \end{bmatrix} \end{aligned} \quad (17)$$

or

$$\begin{aligned} \tilde{A}(z) &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ A(z) &= \frac{1}{256} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ z^{-1} & 5z^{-1} \end{bmatrix}. \end{aligned} \quad (18)$$

The corresponding filters are given in Table I. Note that the filters are not normalized.

### III. CONCLUSION AND FURTHER RESEARCH

We have developed a factorization approach to obtain linear phase double density wavelet filters for a special case where the determinant of  $Q(z)$  is a real number. Eventhough we haven't reported the results, we have observed that our factorization method can be used to find linear phase wavelet filters for the cases (4, 3), (5, 4), etc.

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