

# Design of Double Density Wavelet Filter Banks

Ashoka Jayawardena

School of Mathematical and Computer Sciences,

University of New England,

Armidale, NSW, 2351,

Email: ashoka@turing.une.edu.au

**Abstract**—We look at the design of oversampled filter banks and the resulting framelets. The framelets we will design, will have improved shift invariant properties over *decimated wavelet transform*. Shift invariance has applications in many areas particularly denoising and coding and compression. Our contribution here is on filter bank completion. We will develop factorization methods to find wavelet filters from given scaling filters.

We will look at a special class of framelets from a filter bank perspective, in that we will design double density filter banks (DDFB's) as shown in Figure 1. We denote the z-transform of a sequence  $h(\cdot)$  as  $H(z)$  and its Fourier transform as  $H^f(\omega)$ . Now, for the perfect reconstruction, i.e.  $Y(z) = X(z)$ , it must be necessary that

$$H_0(z)\tilde{H}_0(z) + H_1(z)\tilde{H}_1(z) + H_2(z)\tilde{H}_2(z) = 2, \quad (1)$$

$$H_0(z)\tilde{H}_0(-z) + H_1(z)\tilde{H}_1(-z) + H_2(z)\tilde{H}_2(-z) = 0. \quad (2)$$

Alternatively we can write the above perfect reconstruction conditions in the polyphase domain. Given the following polyphase matrices:

$$\tilde{H}(z) = \begin{bmatrix} \tilde{H}_{00}(z) & \tilde{H}_{01}(z) \\ \tilde{H}_{10}(z) & \tilde{H}_{11}(z) \\ \tilde{H}_{20}(z) & \tilde{H}_{21}(z) \end{bmatrix}, H(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \\ H_{20}(z) & H_{21}(z) \end{bmatrix} \quad (3)$$

where  $\tilde{H}(z)$  is the type 1 analysis polyphase matrix, and  $H(z)$  is the type 2 synthesis polyphase matrix, we can write the perfect reconstruction conditions as

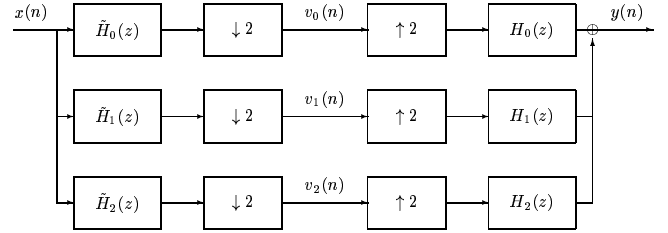


Fig. 1. The double density filter bank

$$[H(z)]^T \tilde{H}(z) = I. \quad (4)$$

## I. NECESSARY CONDITIONS FOR PERFECT RECONSTRUCTION

In the two-band maximally decimated filter banks, for perfect reconstruction it is necessary that the scaling filters,  $H_0(z)$  and  $\tilde{H}_0(z)$ , satisfy the bi-orthogonality constraint,  $H_0(z)\tilde{H}_0(z)[\downarrow 2] = 1$ . Thus most of the designs were dominated to ensure this bi-orthogonality constraint [3], [1], [4]. In the design of double density filter banks we no more have the bi-orthogonality constraint. Thus strictly speaking we do not have bi-orthogonal or orthogonal double density filter banks. But we will design *bi-orthogonal-like* and *orthogonal-like* double density filter banks. We use the term orthogonal-like when the analysis filters and synthesis filters are related via,  $\tilde{H}_i(z) = H_i(z^{-1})$  for  $i \in \{0, 1, 2\}$ . When the filters are not orthogonal-like we say the double density filter bank is bi-orthogonal-like. Thus we can select scaling filters as we like and complete the filter bank such that it satisfy the perfect reconstruction. But vanish-

ing moments for the wavelet filters do impose constraints on the design of scaling filters [2].

## II. DESIGN OF THE SCALING FILTER

We will discuss a technique due to [5]. In that, Selesnick describes a maximally flat symmetric FIR filter which was originally described by Herrman [6]. Let  $P_0(z)$  be the product filter which satisfies the constraints  $P_0^f(0) = 2$ ,  $P_0^{f^{(i)}}(0) = 0$  for  $i \in \{1 \dots 2K_c\}$ ,  $P_0^{f^{(i)}}(\pi) = 0$  for  $i \in \{0 \dots 2K_0\}$ , and  $P_0(z)$  is symmetric, then the product filter is given by

$$P_0(z) = 2 \left( \frac{z + 2 + z^{-1}}{4} \right)^{K_0} \sum_{n=0}^{K_c-1} \binom{K_0 + n - 1}{n} \cdot \left( \frac{-z + 2 - z^{-1}}{4} \right)^n. \quad (5)$$

Now orthogonal-like scaling filters can be obtained by spectral factorization of the product filter while bi-orthogonal-like scaling filters can be obtained by polynomial factorization and appropriate regrouping of the factors.

## III. A POLYPHASE BASED DESIGN APPROACH

As with many problems in signal processing, polyphase constructions substantially simplify constraints. It is the same with double density filter banks. Let the analysis and synthesis wavelet filters are of the form

$$\begin{aligned} \tilde{H}_1(z) &= (1-z)^{\tilde{K}_h} \tilde{A}_1(z), & \tilde{H}_2(z) &= (1-z)^{\tilde{K}_h} \tilde{A}_2(z), \\ H_1(z) &= (1-z^{-1})^{K_h} A_1(z), & H_2(z) &= (1-z^{-1})^{K_h} A_2(z). \end{aligned} \quad (6)$$

Let the type 1 polyphase matrix of analysis wavelet filters and type 2 polyphase matrix of synthesis wavelet filters are given by

$$\tilde{H}_h(z) = \begin{bmatrix} \tilde{H}_{10}(z) & \tilde{H}_{11}(z) \\ \tilde{H}_{20}(z) & \tilde{H}_{21}(z) \end{bmatrix}, \quad H_h(z) = \begin{bmatrix} H_{10}(z) & H_{11}(z) \\ H_{20}(z) & H_{21}(z) \end{bmatrix}.$$

Then from equation 4 it is easy to see that

$$H_h^T(z) \tilde{H}_h(z) = I - \begin{bmatrix} H_{00}(z) \\ H_{01}(z) \end{bmatrix} \begin{bmatrix} \tilde{H}_{00}(z) & \tilde{H}_{01}(z) \end{bmatrix}. \quad (7)$$

We can obtain a simplified expression for both  $\tilde{H}_h(z)$  and  $H_h(z)$ . Note that the type 1 polyphase vector of  $(1-z)S(z)$

can be written as  $\begin{bmatrix} 1 & -1 \\ -z & 1 \end{bmatrix} S_p(z)$  where  $S_p(z)$  is the type 1 polyphase vector of  $S(z)$ . Thus the type 1 polyphase vector  $\begin{bmatrix} \tilde{H}_{10}(z) \\ \tilde{H}_{11}(z) \end{bmatrix}$  can be written in the following form.

$$\begin{bmatrix} \tilde{H}_{10}(z) \\ \tilde{H}_{11}(z) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -z & 1 \end{bmatrix}^{\tilde{K}_h} \tilde{B}_1(z).$$

Similarly

$$\begin{bmatrix} \tilde{H}_{20}(z) \\ \tilde{H}_{21}(z) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -z & 1 \end{bmatrix}^{\tilde{K}_h} \tilde{B}_2(z).$$

Thus

$$\tilde{H}_h(z) = \tilde{A}(z) \begin{bmatrix} 1 & -z \\ -1 & 1 \end{bmatrix}^{\tilde{K}_h} \quad (8)$$

where  $\tilde{A}(z) = \begin{bmatrix} \tilde{B}_1^T(z) \\ \tilde{B}_2^T(z) \end{bmatrix}$ . Similarly

$$H_h(z) = A(z) \begin{bmatrix} 1 & -z^{-1} \\ -1 & 1 \end{bmatrix}^{K_h}. \quad (9)$$

Let

$$P(z) = \begin{bmatrix} 1 & 1 \\ z^{-1} & 1 \end{bmatrix}^{K_h} \begin{bmatrix} I - \begin{bmatrix} H_{00}(z) \\ H_{01}(z) \end{bmatrix} \begin{bmatrix} \tilde{H}_{00}(z) & \tilde{H}_{01}(z) \end{bmatrix} \\ \begin{bmatrix} 1 & z \\ 1 & 1 \end{bmatrix}^{\tilde{K}_h} \end{bmatrix} \quad (10)$$

and  $Q(z) = \frac{1}{(1-z^{-1})^{K_h} (1-z)^{\tilde{K}_h}} P(z)$ . Since

$$\begin{aligned} \begin{bmatrix} 1 & -z \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & z \\ 1 & 1 \end{bmatrix} &= \begin{bmatrix} 1-z & 0 \\ 0 & 1-z \end{bmatrix}, \\ \begin{bmatrix} 1 & 1 \\ z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -z^{-1} & 1 \end{bmatrix} &= \begin{bmatrix} 1-z^{-1} & 0 \\ 0 & 1-z^{-1} \end{bmatrix} \end{aligned} \quad (11)$$

we get

$$A^T(z) \tilde{A}(z) = Q(z). \quad (12)$$

Many factorization methods of Laurent polynomial matrices are crucially dependent on the determinant of the matrix. Fortunately, for a good subclass of double density filter banks, we

have a simplified result for the determinant of  $Q(z)$ . We assume that the number of vanishing moments of analysis high pass filters and synthesis high pass filters are equal, i.e.  $K_h = \tilde{K}_h = M$ . It can be seen that the determinant of  $H_h^T(z)\tilde{H}_h(z)$  is given by  $D(z) = 1 - H_{00}(z)\tilde{H}_{00}(z) - H_{01}(z)\tilde{H}_{01}(z)$ . Since  $Q(z)$  is FIR and both  $\begin{bmatrix} 1 & -z \\ -1 & 1 \end{bmatrix}^M$  and  $\begin{bmatrix} 1 & -1 \\ -z^{-1} & 1 \end{bmatrix}^M$  are factors of  $H_h^T(z)\tilde{H}_h(z)$ , it must be that  $(1-z)^M(1-z^{-1})^M$  is a factor of  $D(z)$ . We have the following Lemma.

*Lemma 1:* Let the high pass filters has  $M$  number of vanishing moments each and the low pass filters,  $H_0$  and  $\tilde{H}_0$ , have the equal regularity,  $K$ , and given by

$$H_0(z)\tilde{H}_0(z) = 2 \left( \frac{z+2+z^{-1}}{4} \right)^K \sum_{n=0}^{M-1} \binom{K+n-1}{n} \cdot \left( \frac{-z+2-z^{-1}}{4} \right)^n. \quad (13)$$

Then the determinant of  $Q(z)$  is a real number when,

$$M = \left\lfloor \frac{K+M-1}{2} \right\rfloor. \quad (14)$$

*Proof:* Please refer [2]. ■

The above lemma covers some useful number of double density filter banks irrespective whether they are bi-orthogonal-like or orthogonal-like such as  $(K, M) = (2, 1), (3, 1), (3, 2), (4, 2), (4, 3), (5, 3), (5, 4), (6, 4), (6, 5)$  etc.

#### A. Factorizations for Orthogonal-like DDFB's

Note that the polyphase matrix of the orthogonal-like DDFB is a  $2 \times 3$  lossless system [7]. Selesnick [5] designs the high pass filters indirectly by first designing a  $3 \times 3$  lossless system and then extracting only the first two rows to form a  $2 \times 3$  lossless system. We provide a more direct method for a class of DDFB's by factorizing  $Q(z)$  when  $\det(Q(z))$  is a real number (as in Lemma 1).

*Lemma 2:* Let

$$Q_0(z) = P_{0,k}^T z^k + P_{0,k-1}^T z^{k-1} + \dots + P_{0,0} + \dots + P_{0,k-1} z^{-(k-1)} + P_{0,k} z^{-k}$$

and assume that  $\det(Q_0(z))$  is a nonzero real number. Let

$$Q_0^{-1}(z) = R_{0,\tilde{k}}^T z^{\tilde{k}} + R_{0,\tilde{k}-1}^T z^{\tilde{k}-1} + \dots + R_{0,0} + \dots + R_{0,\tilde{k}-1} z^{-(\tilde{k}-1)} + R_{0,\tilde{k}} z^{-\tilde{k}}$$

and  $A_0(z) = R_{0,\tilde{k}} + zR_{0,\tilde{k}}^T$  and  $A_0(z)$  is nonsingular. Then for some  $Q_1(z)$ ,

$$Q_0(z) = [A_0(z^{-1})]^{-T} Q_1(z) [A_0(z)]^{-1} \quad (15)$$

where

$$Q_1(z) = P_{1,k-1}^T z^{k-1} + P_{1,k-2}^T z^{k-2} + \dots + P_{1,0} + \dots + P_{1,k-2} z^{-(k-2)} + P_{1,k-1} z^{-(k-1)}.$$

*Proof:* Please refer [2]. ■

Assuming each degree reduction step is invertible, repeating the process given in Lemma 2, we could write

$$A_k^T(z^{-1}) \dots A_0^T(z^{-1}) Q_0(z) A_0(z) \dots A_k(z) = Q_{k+1} \quad (16)$$

where  $Q_{k+1}$  is a constant symmetric matrix (i.e.  $Q_{k+1} = Q_{k+1}^T$ ). Then  $Q_{k+1}$  is orthogonally diagonalizable and let  $Q_{k+1} = A_{k+1}^T A_{k+1}$ . However we require the eigenvalues of  $Q_{k+1}$  be positive. Then we could write

$$A^T(z) = A_0^{-T}(z^{-1}) \dots A_k^{-T}(z^{-1}) A_{k+1}^T. \quad (17)$$

We have one degree of freedom in the factorization as given by

$$A^T(z) = A_0^{-T}(z^{-1}) \dots A_k^{-T}(z^{-1}) A_{k+1}^T \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \quad (18)$$

Such parameterization is used by Selesnick [5] to achieve near shift invariance among the wavelet functions.

*Example 1:* Consider the (2,1) orthogonal-like DDFB with  $H_0(z) = \sqrt{2} \left( \frac{1+z}{2} \right)^2$  and  $\tilde{H}_0(z) = \sqrt{2} \left( \frac{1+z^{-1}}{2} \right)^2$ . We get

$$Q(z) = \begin{bmatrix} \frac{3}{8} & \frac{z}{8} \\ \frac{z^{-1}}{8} & \frac{3}{8} \end{bmatrix} \text{ which leads to}$$

$$A^T(z) = \left[ \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + z^{-1} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \right]^{-T} \begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{2}} \end{bmatrix}.$$

The synthesis high pass filters are given by  $H_h^T(z) = \begin{bmatrix} 1 & -1 \\ -z^{-1} & 1 \end{bmatrix} A^T(z)$ . This gives  $H_1(z) = \frac{1}{2} - \frac{1}{2}z^2$  and  $H_2(z) = \frac{2\sqrt{2}}{4} - \frac{2\sqrt{2}}{2}z + \frac{2\sqrt{2}}{4}z^2$ .

Clearly Lemma 2 is not always applicable since there is no guarantee that the degree reduction step is invertible. But so far in all the examples I have computed, I have not run into this problem. Whether it is that we have just been lucky or that we haven't uncovered some of the hidden results, is not yet clear to me. But when we generalize double density into multiple density case, we will realize that we have been lucky in the double density case.

#### IV. CONCLUSION AND FURTHER RESEARCH

We have developed a factorization approach to obtain double density wavelet filters for a special case where the determinant of transfer polyphase matrix is a real number. We have analytically obtained example filters for with small number of filter coefficients. However, optimization of such filters was left undone since it can be done similarly as in Selesnick's [5].

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