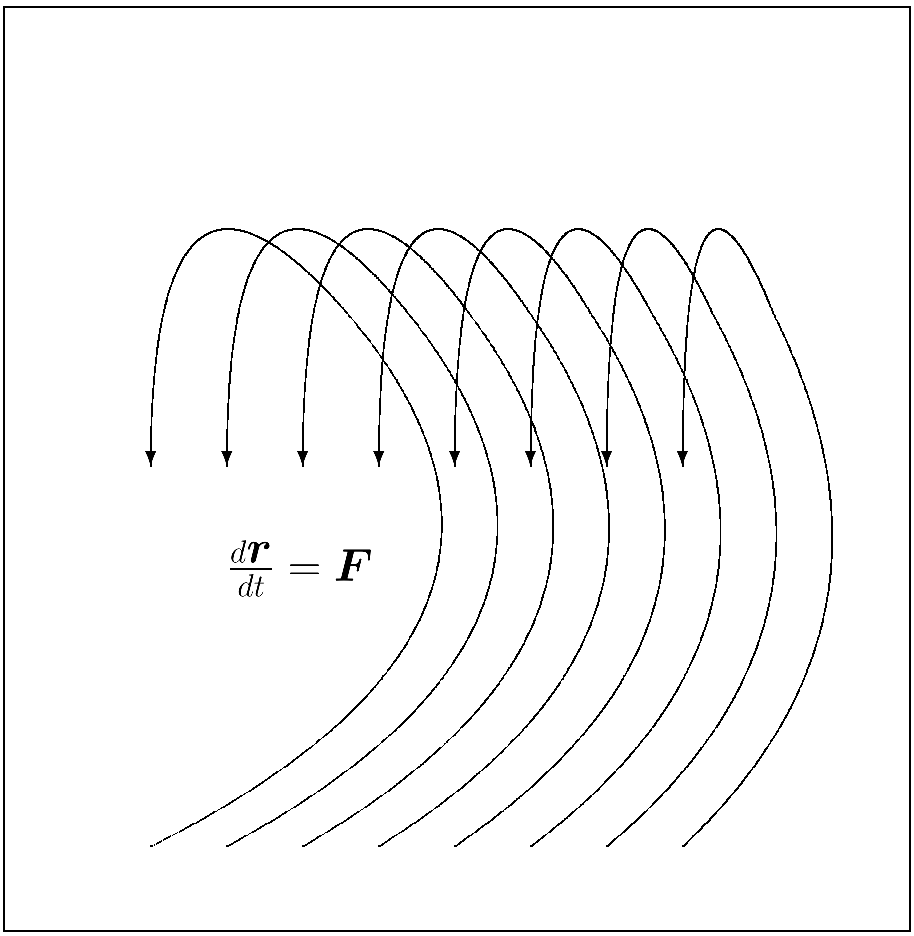
AMTH246

Mathematical Methods In The Sciences I



Contents

1 Preface

This is a unit in applied mathematics. The emphasis here will be on developing a set of mathematical techniques and skills which can be *applied* to various "real world" models. The first half of the unit develops the multi-variable differential and integral calculus needed to tackle such subjects as electromagnetism and fluid dynamics. The second half of the unit is divided into two sections. The first, is devoted to applications of the mathematics to fluid flow problems, fluid dynamics. The second section is an introduction to Fourier series.

A second year unit such as this is offered by most universities around the world. The topics covered are seen as part of the essential training of any engineer, physicist or applied mathematician. Without this background it would be impossible to appreciate some of the more modern areas of mathematical physics.

The stress here is on doing mathematics. It is essential that you do as many exercises and examples as you can. The "Tutorial Booklet" contains many problems. Each tutorial set covers about two lectures, you should try to complete one of these sets for each pair of lectures you study. These tutorial problems are for your own private study, you should try as many of them as you reasonably can. A seperate Solutions booklet is provided, however it is important to attempt the problems 'on your own' in the first instance.

This is a solid unit. The work and problems can be quite demanding. But, I think there are many rewards. There is great satisfaction in applying successfully your mathematical skills to various physical models.

I would also ask you to provide me with any relevant comments, criticisms, suggestions or corrections which you may have about the unit content, structure or administration. We need such feedback.

I am sure you will find this unit intellectually stimulating and enjoyable.

Best wishes,

Dr Tim Dalby

UNE

2 Lecture Schedule

The following dates indicate the week in which the on campus lectures are held. Students on campus have two lectures, one tutorial and one computer laboratory session per week. These dates can serve as a rough guide for your studies.

Lectures 1 and 2	July 23
Lectures 3 and 4	July 30
Lectures 5 and 6	August 6
Lectures 7 and 8	August 13
Lectures 9 and 10	August 20
Lectures 11 and 12	August 27
Lectures 13 and 14	September 3
Lectures 15 and 16	September 10
Lectures 17 and 18	October 2
Lectures 19 and 20	October 9
Lectures 21 and 22	October 15
Lectures 23 and 24	October 22

3 References

The lecture notes for this unit are complete. Together with the practicals they cover all the assessable material. Of course it is always a good idea to refer to other books, if only to get a wider perspective on the subject matter. The following books are suggestions only there are many good reference books in this are, you should be able to find something which suits your tastes.

[1] Adams, R.A. *Calculus of Several Variables*, Second Edition, Addison-Wesley.

[2] Anton, H. *Calculus*, Fourth Edition, John Wiley and Sons.

[3] Chorin, A.J., Marden, J.E. *Mathematical Introduction to Fluid Mechanics*, Springer-Verlag.

[4] Chorlton, F. *Textbook of Fluid Dynamics*, D. Van Norstrand.

[5] Folland, G. *Fourier Analysis and its Applications*, Wadsworth and Brooks/Cole.

[6] Paterson, A.R. *A First Course in Fluid Dynamics*, Cambridge University Press.

[7] Riley, K.F. *Mathematical Methods for the Physical Sciences*, Cambridge University Press.

[8] Schey, H.M. *Div grad curl and all that*, Second Edition, W.W. Norton and Company.

Unit Section	Reference
I Vector Differential Operators	[1] chpt 7, [2] chapt 18, [4] chapt 1, [7], [8].
II The Integral Theorems	[1] chpt 7, [2] chapt 8, [7], [8].
III Variation In Time	[4] chpt 2.
IV Applications To Fluid Mechanics	[3], [4], [6].
V Fourier Series	[5], [7]

I Vector Differential Operators

4 Lecture 1

Introduction

In this section we want to describe a set of differential operators which can be used to describe "rates of change" for functions and vector functions defined in three dimensional space. As we will see below these operators are associated with the various forms of vector multiplication: scalar, "dot" and "cross". Vector differential operators give us a set of natural calculus tools which exploit the vector space character of $\mathbb{R}^3$.

You will need to be reasonably familiar with vector notation and conventions, MATH101 and PMTH212, and the material on multivariable calculus, PMTH212.

The mathematical objects we will be studying will be functions, vectors and vector valued functions (i.e. vector fields). In the applications section of the course we will associate these quantities with physically measurable variables, density, pressure and velocity, for example.

We will denote functions defined on a subset, Ω of three dimensional space, $\mathbb{R}^3$, as

$$\psi = \psi(x, y, z)\,, \text{or}\,,\ \ \psi = \psi(\boldsymbol{r}),$$

where $\boldsymbol{r}$ is the position vector, $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. We will use bold face type to indicate vectors, for handwritten vectors it is conventional to use the printer's notation $\vec{r}$. A function is said to be C^n if it has continuous derivatives up to and including order n.

A vector valued function, or vector field, is a vector whose components at each point in its region of definition are determined by three functions. That is

$$\boldsymbol{F} = F_1(x, y, z)\boldsymbol{i} + F_2(x, y, z)\boldsymbol{j} + F_3(x, y, z)\boldsymbol{k},$$

where F_1, F_2 and F_3 are the three functions.
For example, $\boldsymbol{r}$ is a vector field. At each point in $\mathbb{R}^3$ $\boldsymbol{r}$ determines a vector, e.g. at $(x, y, z) = (1, 0, 1)$, $\boldsymbol{r}$ is $\boldsymbol{i}+\boldsymbol{k}$. A vector field is said to be C^n if each of its components is a C^n function.

We now introduce the first vector differential operator, in fact this operator is the basic building block from which all our vector differential operators will be constructed.

The Gradient

Suppose $\psi = \psi(x, y, z)$ is a C^1 function on some domain Ω of $\mathbb{R}^3$. To describe the way ψ changes as we move through Ω we will need to calculate the three quantities

$$\frac{\partial \psi}{\partial x}, \ \frac{\partial \psi}{\partial y}, \ \frac{\partial \psi}{\partial z}.$$

The fact that there are three components, the x, y and z derivatives of ψ, suggests that we have a vector field. We do and it is called the *gradient* of ψ.

$$\diamond\underline{\text{Gradient}}, \quad \text{grad}(\psi) \equiv \boldsymbol{\nabla}\psi = \frac{\partial \psi}{\partial x}\boldsymbol{i} + \frac{\partial \psi}{\partial y}\boldsymbol{j} + \frac{\partial \psi}{\partial z}\boldsymbol{k}.$$

The *vector differential operator*

$$\boldsymbol{\nabla} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

is known as the gradient operator.

♦**Example** Find grad(ψ), for $\psi = xyz + x^2 + y^2 + z^2$.

<u>Solution</u>: We need the three derivatives of ψ

$$\frac{\partial \psi}{\partial x} = yz + 2x, \ \frac{\partial \psi}{\partial y} = xz + 2y, \ \frac{\partial \psi}{\partial z} = xy + 2z$$

so that $\text{grad}\psi = \boldsymbol{\nabla}\psi = (yz + 2x)\boldsymbol{i} + (xz + 2y)\boldsymbol{j} + (xy + 2z)\boldsymbol{k}$.

□<u>Example</u>. Prove that if $\psi = \psi(\boldsymbol{r})$ is a differentiable function then $\boldsymbol{\nabla}\psi$ is normal to the level surfaces of ψ.

<u>Solution</u>: Consider a neighbourhood, N, of a level surface, $\psi = C$ (C a constant), and let $\mathcal{C}$ be any differentiable parametric curve defined in N and lying in the level surface. Suppose $\mathcal{C}$ is defined as

$$\begin{aligned} x &= x(t) \\ y &= y(t) \\ z &= z(t). \end{aligned}$$

Then $\psi(x(t), y(t), z(t)) = C$. So,

$$\frac{d}{dt}\left[\psi(x(t), y(t), z(t))\right] = 0.$$

Using the chain rule we arrive at

$$\frac{\partial \psi}{\partial x}\frac{dx}{dt} + \frac{\partial \psi}{\partial y}\frac{dy}{dt} + \frac{\partial \psi}{\partial z}\frac{dz}{dt} = 0.$$

This can be written as $(\boldsymbol{\nabla}\psi)\boldsymbol{.}\boldsymbol{v} = 0$ where $\boldsymbol{v} = \dot{x}(t)\boldsymbol{i} + \dot{y}(t)\boldsymbol{j} + \dot{z}(t)\boldsymbol{k}$ is the tangent vector to $\mathcal{C}$. Hence, $\boldsymbol{\nabla}\psi$ is perpendicular to any curve in $\psi = C$ – recall that two non-zero vectors are orthogonal iff their dot product vanishes.
We see that $\boldsymbol{\nabla}\psi$ is normal to every curve in the level surface and we may conclude that it is normal to the level surface itself.

The rate of change of a differentiable function, in a given direction $\boldsymbol{e}$, is called its *directional derivative.*

$$\diamond \quad D_{\hat{\boldsymbol{e}}}\psi(x_0, y_0, z_0) = (\hat{\boldsymbol{e}}\boldsymbol{.}\boldsymbol{\nabla}\psi) \mid_{(x_0,y_0,z_0)},$$

is the directional derivative of ψ, in the direction $\boldsymbol{e}$, evaluated at the point (x_0, y_0, z_0). Here, $\hat{\boldsymbol{e}} = \boldsymbol{e}/\mid \boldsymbol{e} \mid$ is the unit vector in the $\boldsymbol{e}$–direction; $\mid \boldsymbol{e} \mid= e$ is the length of $\boldsymbol{e}$.

Note that

$$D_{\boldsymbol{i}}\psi = \frac{\partial \psi}{\partial x}, \quad D_{\boldsymbol{j}}\psi = \frac{\partial \psi}{\partial y}, \quad D_{\boldsymbol{k}}\psi = \frac{\partial \psi}{\partial z}.$$

♦**Example** In which direction does the function $f(x, y, z) = xyz$ increase most rapidly and what is this rate of change at $(1, 1, 1)$?

Solution: Recall that the dot product between two vectors can be written as $\boldsymbol{a}\boldsymbol{.}\boldsymbol{b} = ab\cos\theta$, where θ is the angle between them. In terms of the directional derivative we see that $\hat{\boldsymbol{e}}\boldsymbol{.}\boldsymbol{\nabla}f$ will be a maximum when $\boldsymbol{e}$ is in the same direction as the gradient of f. So the direction of maximal rate of change of f is just $\boldsymbol{\nabla}f$, i.e. the direction of the normal to the level surfaces of f. To calculate this rate of change we need the directional derivative of f in this direction. Now

$$\boldsymbol{\nabla}f = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}.$$

At $(1, 1, 1)$ we have

$$\boldsymbol{\nabla}f \mid_{(1,1,1)}= \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k},$$

this is the direction of maximal rate of change at $(1, 1, 1)$, the unit vector in this direction is $\hat{\boldsymbol{e}} = (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})/\sqrt{3}$ and the required rate of change is

$$D_{\hat{\boldsymbol{e}}}f = (\boldsymbol{\nabla}f)\boldsymbol{.}\hat{\boldsymbol{e}} \mid_{(1,1,1)}= (\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})\boldsymbol{.}\frac{(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})}{\sqrt{3}} = \sqrt{3}.$$

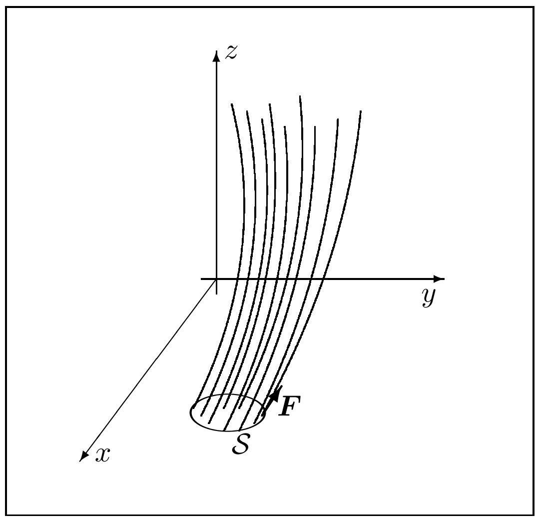
The Divergence

Let $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$ be a C^1 vector field. Information about the way $\boldsymbol{F}$ changes on its domain is contained in nine partial derivatives, three for each component. One particularly important grouping of these derivatives is the *divergence*. The divergence of a vector field is one of three possible operators one can construct, in a natural manner, from the gradient operator. The action of multiplication by a scalar in a vector space gives rise to the gradient of a scalar, $\boldsymbol{\nabla}\psi$, the dot product of a pair of vectors gives rise to the divergence of a vector field.

$\diamond$ The Divergence $\quad \mathrm{div}\boldsymbol{F} = \boldsymbol{\nabla.F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}.$

Note that div$\boldsymbol{F}$ is a scalar. Loosely speaking, the divergence measures how $\boldsymbol{F}$ "spreads". Through each point in a small surface in $\mathbb{R}^3$ there is a unique curve with $\boldsymbol{F}$ as tangent vector (as we will see later these curves are called field lines or integral curves).

Then div$\boldsymbol{F}$ measures the rate at which these curves expand away from each other (if the divergence is positive) or contract toward each other (negative divergence). The diagram shows the integral curves for $\boldsymbol{F}$ expanding from the surface $\mathcal{S}$. The divergence of $\boldsymbol{F}$ in a neighbourhood of $\mathcal{S}$ would be positive.



♦**Example** Find div$\boldsymbol{r}$

Solution: From the definition

$$\mathrm{div}\boldsymbol{r} \equiv \boldsymbol{\nabla.r} = \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(z)}{\partial z} = 3.$$

So the integral curves of $\boldsymbol{r}$, i.e. the curves with $\boldsymbol{r}$ as tangent vector, spread out or expand in the direction of $\boldsymbol{r}$. This is quite clear if you draw a picture, the integral curves of $\boldsymbol{r}$ are just the radial straight lines through the origin.

♦**Example** If ψ is a C^1 scalar field and $\boldsymbol{F}$ is a C^1 vector field show that

$$\boldsymbol{\nabla.}(\psi\boldsymbol{F}) = (\boldsymbol{\nabla}\psi)\boldsymbol{.F} + \psi\boldsymbol{\nabla.F}.$$

Solution: From the definition of divergence we have

$$\boldsymbol{\nabla}\boldsymbol{.}(\psi \boldsymbol{F}) = \frac{\partial(\psi F_1)}{\partial x} + \frac{\partial(\psi F_2)}{\partial y} + \frac{\partial(\psi F_3)}{\partial z}$$

Using the product rule for differentiation we get, after rearranging

$$\boldsymbol{\nabla}\boldsymbol{.}(\psi \boldsymbol{F}) = \left(F_1 \frac{\partial \psi}{\partial x} + F_2 \frac{\partial \psi}{\partial y} + F_3 \frac{\partial \psi}{\partial z} \right) + \psi \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right).$$

Finally, from the definition of the "dot" product and the definition of divergence we get the required result.

5 Lecture 2

The Laplacian

Let $\psi = \psi(\boldsymbol{r})$ be a C^2 function on some three dimensional domain. Then $\boldsymbol{\nabla}\psi$ is a C^1 vector field on this domain, so we form its divergence.

$$\begin{aligned}\boldsymbol{\nabla .}(\boldsymbol{\nabla}\psi) &= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z}\right) \\ &= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}.\end{aligned}$$

The second order differential operator appearing on the right is known as the *Laplacian.*

$$\diamond \ \underline{\text{Laplacian}} \equiv \triangle \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

As we have derived it the Laplacian can be applied only to scalar fields, in a later lecture we will show how it can be defined to apply to vector fields.

One of the most important equations of mathematical physics is *Laplace's equation* $\nabla^2\psi = 0$. This equation is found in electrostatics, hydrostatics, Newtonian gravitation and stationary wave and diffusion processes. Laplace 's equation is a second order, linear, partial differential equation. A function satisfying $\nabla^2\psi = 0$ in some region, Ω, is said to be harmonic in Ω. Laplace's equation is linear. This means that if u and v are both harmonic functions, i.e. $\nabla^2 u = 0$ and $\nabla^2 v = 0$, then $\alpha u + \beta v$ is also harmonic for any constants α and β.

♦**Example** For what values of n is r^n an harmonic function?

<u>Solution</u>: First we note a useful result, it's just the chain rule in terms of the gradient. For $f = f(u)$ and $u = u(x, y, z)$ we have

$$\boldsymbol{\nabla} f = f'(u)\boldsymbol{\nabla} u.$$

Applying this result to r^n we have

$$\boldsymbol{\nabla} r^n = nr^{n-1}\boldsymbol{\nabla} r = nr^{n-1}\frac{\boldsymbol{r}}{r} = nr^{n-2}\boldsymbol{r},$$

where we have used

$$\begin{aligned}\boldsymbol{\nabla} r = \boldsymbol{\nabla}\sqrt{x^2+y^2+z^2} &= \frac{x}{r}\boldsymbol{i} + \frac{y}{r}\boldsymbol{j} + \frac{z}{r}\boldsymbol{k} \\ &= \frac{\boldsymbol{r}}{r} \\ &= \hat{\boldsymbol{r}}.\end{aligned}$$

Next, apply the result of the last example of the previous lecture (with $\boldsymbol{F} = \boldsymbol{r}$ and $\psi = nr^{n-2}$) to find $\boldsymbol{\nabla}.(\boldsymbol{\nabla} r^n)$,

$$\begin{aligned}
\nabla^2 r^n &= \boldsymbol{\nabla}.(\boldsymbol{\nabla} r^n) \\
&= \boldsymbol{r}.\boldsymbol{\nabla}(nr^{n-2}) + nr^{n-2}\boldsymbol{\nabla}.\boldsymbol{r} \\
&= n(n-2)r^{n-4}\boldsymbol{r}.\boldsymbol{r} + 3nr^{n-2} \\
&= [n(n-2)+3n]r^{n-2} \\
&= n(n+1)r^{n-2}.
\end{aligned}$$

Consequently, r^n is harmonic in $\mathbb{R}^3$ (excluding $r = 0$) iff $n = 0$ or $n = -1$, so that 1 and $1/r$ are our solutions. In fact, by linearity, we know that $\alpha + \beta/r$ is harmonic.

The Curl

The last of our vector differential operators results from forming the cross product of the gradient operator with a vector field.

Suppose $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$ is a C^1 vector field then the *curl* of $\boldsymbol{F}$ is defined to be

$$\diamond \quad \mathrm{curl}\boldsymbol{F} \equiv \boldsymbol{\nabla} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}.$$

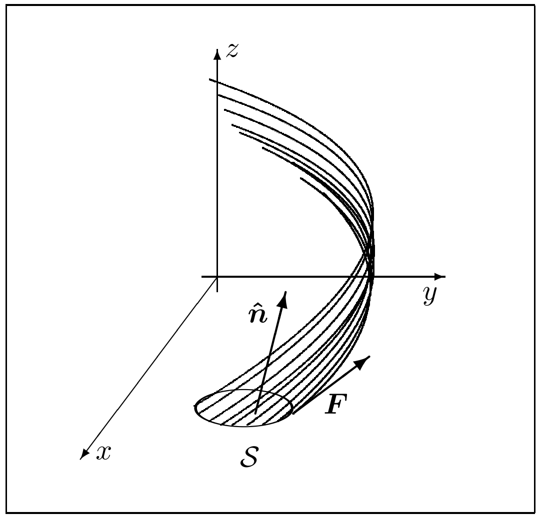
These formulae for curl follow from $\boldsymbol{\nabla} \times \boldsymbol{F}$ if you multiply out the expression term by term using

$$\begin{aligned}
\boldsymbol{i} \times \boldsymbol{j} &= \boldsymbol{k} \\
\boldsymbol{i} \times \boldsymbol{k} &= -\boldsymbol{j} \\
\boldsymbol{j} \times \boldsymbol{k} &= \boldsymbol{i}.
\end{aligned}$$

You should try this as an exercise!

One of the older names for the curl (no longer in use) was "rotation", abbreviated to rot$\boldsymbol{F}$, this descriptive term indicates the mathematical meaning of curl$\boldsymbol{F}$.

The curl of a vector field measures the twist of the vector $\boldsymbol{F}$ – it measures how much the field lines twist about one another (think of a barber's pole). The diagram shows the field lines for $\boldsymbol{F}$ twisting in the positive sense (anti-clockwise) about the normal to the surface $\mathcal{S}$.



♦**Example** Find the curl of the vector field $\boldsymbol{v} = -y\boldsymbol{i} + x\boldsymbol{j}$

<u>Solution</u>: Using the determinant form of the definition we have

$$\begin{aligned} \text{curl}\boldsymbol{v} \equiv \boldsymbol{\nabla} \times \boldsymbol{v} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} \\ &= \left(\frac{\partial(0)}{\partial y} - \frac{\partial(x)}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial(0)}{\partial x} - \frac{\partial(-y)}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\right)\boldsymbol{k} \\ &= 2\boldsymbol{k}. \end{aligned}$$

♦**Example** Show that $\boldsymbol{\nabla} \times \boldsymbol{r} = 0$.

<u>Solution</u>: Using the determinant definition we have

$$\begin{aligned} \text{curl}\boldsymbol{r} \equiv \boldsymbol{\nabla} \times \boldsymbol{r} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\ &= \left(\frac{\partial(z)}{\partial y} - \frac{\partial(y)}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial(z)}{\partial x} - \frac{\partial(x)}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial(y)}{\partial x} - \frac{\partial(x)}{\partial y}\right)\boldsymbol{k} \\ &= \boldsymbol{0}. \end{aligned}$$

♦**Example** Prove that for a C^2 function ψ

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) = 0.$$

<u>Solution</u>: A simple calculation

$$\text{curl}(\boldsymbol{\nabla}\psi) \equiv \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\psi}{\partial x} & \frac{\partial\psi}{\partial y} & \frac{\partial\psi}{\partial z} \end{vmatrix}$$

$$
\begin{aligned}
&= \left[\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial y}\right)\right]\boldsymbol{i} - \left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial x}\right)\right]\boldsymbol{j} + \\
&\quad \left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)\right]\boldsymbol{k} \\
&= \mathbf{0}.
\end{aligned}
$$

6 Lecture 3

Identities Involving Gradient, Divergence and Curl

There are many identities connecting our vector differential operators, we have collected nine of the most useful of them in the following theorem.

◇ Theorem: *Let ψ and η be scalar fields and $\boldsymbol{F}$ and $\boldsymbol{G}$ be vector fields. We assume that all these fields are sufficiently differentiable that the partial derivatives in the nine identities below exist and are continuous. Then the following identities hold*

1. $\boldsymbol{\nabla}(\psi\eta) = \psi\boldsymbol{\nabla}\eta + \eta\boldsymbol{\nabla}\psi$.

2. $\boldsymbol{\nabla}.(\psi\boldsymbol{F}) = \psi(\boldsymbol{\nabla}.\boldsymbol{F}) + (\boldsymbol{F}.\boldsymbol{\nabla})\psi$.

3. $\boldsymbol{\nabla}\times(\psi\boldsymbol{F}) = (\boldsymbol{\nabla}\psi\times\boldsymbol{F}) + \psi(\boldsymbol{\nabla}\times\boldsymbol{F})$.

4. $\boldsymbol{\nabla}.(\boldsymbol{F}\times\boldsymbol{G}) = (\boldsymbol{\nabla}\times\boldsymbol{F}).\boldsymbol{G} - \boldsymbol{F}.(\boldsymbol{\nabla}\times\boldsymbol{G})$.

5. $\boldsymbol{\nabla}\times(\boldsymbol{F}\times\boldsymbol{G}) = (\boldsymbol{\nabla}.\boldsymbol{G})\boldsymbol{F} + (\boldsymbol{G}.\boldsymbol{\nabla})\boldsymbol{F} - (\boldsymbol{\nabla}.\boldsymbol{F})\boldsymbol{G} - (\boldsymbol{F}.\boldsymbol{\nabla})\boldsymbol{G}$.

6. $\boldsymbol{\nabla}(\boldsymbol{F}.\boldsymbol{G}) = \boldsymbol{F}\times(\boldsymbol{\nabla}\times\boldsymbol{G}) + \boldsymbol{G}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) + (\boldsymbol{F}.\boldsymbol{\nabla})\boldsymbol{G} + (\boldsymbol{G}.\boldsymbol{\nabla})\boldsymbol{F}$.

7. $\boldsymbol{\nabla}.(\boldsymbol{\nabla}\times\boldsymbol{F}) = 0$.

8. $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) = \boldsymbol{0}$.

9. $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{F}) - \nabla^2\boldsymbol{F}$.

Proof
1. Follows as an immediate consequence of the product rule for differentiation. Try it!
2. Proof given in example of Lecture 1.
3. This is again just a slightly disguised form of the product rule.
4. First we note that

$$\begin{aligned}
(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}).(\boldsymbol{F}\times\boldsymbol{G}) &= \boldsymbol{i}.\left[\frac{\partial}{\partial x}(\boldsymbol{F}\times\boldsymbol{G})\right] + \cdots \\
&= \boldsymbol{i}.\left(\frac{\partial \boldsymbol{F}}{\partial x}\times\boldsymbol{G} + \boldsymbol{F}\times\frac{\partial \boldsymbol{G}}{\partial x}\right) + \cdots \\
&= \boldsymbol{i}.\frac{\partial \boldsymbol{F}}{\partial x}\times\boldsymbol{G} + \boldsymbol{i}.\boldsymbol{F}\times\frac{\partial \boldsymbol{G}}{\partial x} + \cdots \\
&= \boldsymbol{G}.\boldsymbol{i}\times\frac{\partial \boldsymbol{F}}{\partial x} + \boldsymbol{F}.\frac{\partial \boldsymbol{G}}{\partial x}\times\boldsymbol{i} + \cdots \\
&= \boldsymbol{G}.\boldsymbol{i}\times\frac{\partial \boldsymbol{F}}{\partial x} - \boldsymbol{F}.\boldsymbol{i}\times\frac{\partial \boldsymbol{G}}{\partial x} + \cdots.
\end{aligned}$$

Here $\cdots$ represent the terms of similar type (those involving $\boldsymbol{j}$ and $\boldsymbol{k}$). In the second line the product rule has been used and in the fourth line the following vector identity has been used on the scalar triple products

$$\boldsymbol{a}.\boldsymbol{b} \times \boldsymbol{c} = \boldsymbol{c}.\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{b}.\boldsymbol{c} \times \boldsymbol{a}.$$

Finally, we note that $\boldsymbol{i}\times(\partial \boldsymbol{F}/\partial x)$ is just the $\boldsymbol{i}$ "part" of $\boldsymbol{\nabla} \times \boldsymbol{F}$ (similarly for the expression in $\boldsymbol{G}$). The result now follows.

5. Now,

$$\boldsymbol{F} \times \boldsymbol{G} = (F_2G_3 - F_3G_2)\boldsymbol{i} + (F_3G_1 - F_1G_3)\boldsymbol{j} + (F_1G_2 - F_2G_1)\boldsymbol{k}.$$

Calculating the $\boldsymbol{i}$-component of the curl of this expression gives

$$\frac{\partial}{\partial y}(F_1G_2 - F_2G_1) - \frac{\partial}{\partial z}(F_3G_1 - F_1G_3).$$

The $\boldsymbol{i}$-component of the right hand side of **5** is

$$(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} + \frac{\partial G_3}{\partial z})F_1 + (G_1\frac{\partial}{\partial x} + G_2\frac{\partial}{\partial y} + G_3\frac{\partial}{\partial z})F_1 - (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z})G_1 - (F_1\frac{\partial}{\partial x} + F_2\frac{\partial}{\partial y} + F_3\frac{\partial}{\partial z})G_1.$$

With some effort(!) and many uses of the product rule one shows that these two expressions are in fact the same. The result then follows by observing that the $\boldsymbol{j}$ and $\boldsymbol{k}$ components will follow exactly the same pattern.

6. Firstly we note that

$$\boldsymbol{F}.\boldsymbol{G} = F_1G_1 + F_2G_2 + F_3G_3,$$

so the $\boldsymbol{i}$-component of the left hand side is

$$\frac{\partial}{\partial x}\left(F_1G_1 + F_2G_2 + F_3G_3\right).$$

Now look at the $\boldsymbol{i}$-component of the right hand side. It is the sum of the following four terms

$$\boldsymbol{i}\text{-component of } \boldsymbol{F}\times(\boldsymbol{\nabla} \times \boldsymbol{G}) = F_2\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) - F_3\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right),$$
$$\boldsymbol{i}\text{-component of } \boldsymbol{G}\times(\boldsymbol{\nabla} \times \boldsymbol{F}) = G_2\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - G_3\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right),$$
$$\boldsymbol{i}\text{-component of } (\boldsymbol{F}.\boldsymbol{\nabla})\boldsymbol{G} = F_1\frac{\partial G_1}{\partial x} + F_2\frac{\partial G_1}{\partial y} + F_3\frac{\partial G_1}{\partial z},$$
$$\boldsymbol{i}\text{-component of } (\boldsymbol{G}.\boldsymbol{\nabla})\boldsymbol{F} = G_1\frac{\partial F_1}{\partial x} + G_2\frac{\partial F_1}{\partial y} + G_3\frac{\partial F_1}{\partial z}.$$

Adding these four contributions gives, after using the product rule, the same result as the $\boldsymbol{i}$-component of the left hand side (calculated above). For example, if we take all the terms involving x derivatives from the four contributions we have

$$F_2\frac{\partial G_2}{\partial x} + F_3\frac{\partial G_3}{\partial x} + G_2\frac{\partial F_2}{\partial x} + G_3\frac{\partial F_3}{\partial x} + F_1\frac{\partial G_1}{\partial x} + G_1\frac{\partial F_1}{\partial x},$$

which becomes after using the product rule

$$\frac{\partial}{\partial x}\left(F_1G_1 + F_2G_2 + F_3G_3\right).$$

Similar results are obtained for the y and z derivative terms. So the $\boldsymbol{i}$-components of the left and right sides of **6** are equal. The $\boldsymbol{j}$ and $\boldsymbol{k}$ components can be worked in exactly the same manner and our result follows.

7. Exercise!

8. See example in Lecture 2.

9. See Assignment 1.

7 Lecture 4

Integral Curves

As was mentioned earlier the integral curves of a vector field $\boldsymbol{F}$ are those curves having $\boldsymbol{F}$ as their tangent vector. These curves are given many other names, usually depending on the context – field lines, stream lines or lines of force.

An integral curve will not depend upon the magnitude of $\boldsymbol{F}$ at any given point, it will depend only on the direction of $\boldsymbol{F}$ at that point. Suppose an integral curve for $\boldsymbol{F}$ is given parametrically by

$$\begin{aligned} x &= x(t) \\ y &= y(t) \\ z &= z(t), \end{aligned}$$

or as $\boldsymbol{r} = \boldsymbol{r}(t)$. Then the tangent vector is $d\boldsymbol{r}/dt$ and by definition of our integral curve this must be parallel to $\boldsymbol{F}(\boldsymbol{r}(t))$ for all t. Recall that two vectors are parallel iff they are proportional, so we have

$$\diamond\underline{\text{Integral Curve}} \quad \frac{d\boldsymbol{r}}{dt} = \lambda(t)\boldsymbol{F}(\boldsymbol{r}(t)).$$

Here $\lambda = \lambda(t)$ is the proportionality factor – it can, of course vary from point to point on the integral curve.

⧫**Example** The gravitational force, per unit mass, due to a point mass located at the origin is, in Newtonian theory,

$$\boldsymbol{F} = -\frac{\gamma m}{r^2}\hat{\boldsymbol{r}}, \quad \gamma \text{ a constant.}$$

Find the field lines of $\boldsymbol{F}$.

Solution: On straightforward physical grounds we expect the field lines to be radial lines directed towards the origin – the lines of force. Let's see if this is the consequence of our definitions

$$\frac{d\boldsymbol{r}}{dt} = -\frac{\lambda(t)\gamma m}{r^2}\hat{\boldsymbol{r}}.$$

This vector equation is equivalent to the three component equations

$$\frac{dx}{dt} = -\frac{\lambda\gamma m}{r^2}\frac{x}{r}$$

$$\begin{aligned}\frac{dy}{dt} &= -\frac{\lambda\gamma m}{r^2}\frac{y}{r}\\ \frac{dz}{dt} &= -\frac{\lambda\gamma m}{r^2}\frac{z}{r},\end{aligned}$$

since, $\hat{\boldsymbol{r}} = \frac{x}{r}\boldsymbol{i} + \frac{y}{r}\boldsymbol{j} + \frac{z}{r}\boldsymbol{k}$. There are a number of different approaches one can take to solving these equations – divide the second by the first, separate variables and integrate etc. Perhaps the most elegant is the following. First, we note that we can combine the three equations as follows

$$x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = -\frac{\gamma m\lambda}{r^3}(x^2 + y^2 + z^2).$$

But $x^2 + y^2 + z^2 = r^2$ and $2x dx/dt = d(x^2)/dt$ (with similar expressions in y and z), so

$$\frac{1}{2}\frac{d(r^2)}{dt} = -\frac{\gamma m\lambda}{r}.$$

We rewrite this as

$$-\gamma m\lambda = r^2\frac{dr}{dt}.$$

With this our three original equations can be written as

$$\begin{aligned}\frac{dx}{dt} &= \frac{x}{r}\frac{dr}{dt}\\ \frac{dy}{dt} &= \frac{y}{r}\frac{dr}{dt}\\ \frac{dz}{dt} &= \frac{z}{r}\frac{dr}{dt}.\end{aligned}$$

These equations are easily integrated by separation of variables. For example, we will integrate the x equation. First separate the variables

$$\frac{1}{x}\frac{dx}{dt} = \frac{1}{r}\frac{dr}{dt}.$$

Or $dx/x = dr/r$. Integrating both sides gives $\ln x = \ln r + \text{constant}$ or $x = ar$, where a is a constant. Altogether we get

$$\begin{aligned}x &= ar\\ y &= br\\ z &= cr,\end{aligned}$$

here a, b and c are three constants. Notice that $a^2 + b^2 + c^2 = 1$ (why?). The above equations are of course those of a straight line through the origin – varying the constants gives different lines.

♦**Example** The velocity of a solid rotating about the z-axis with angular velocity $\boldsymbol{\Omega} = \omega \boldsymbol{k}$ is

$$\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r} = \omega(-y\boldsymbol{i} + x\boldsymbol{j}).$$

Find the field lines of $\boldsymbol{v}$.

<u>Solution</u>: This was one of examples on curl from Lecture 2. We now have to find the integral curves for this vector field. The equations for the integral curves, in component form, are

$$\begin{aligned}
\frac{dx}{dt} &= -\lambda\omega y \\
\frac{dy}{dt} &= \lambda\omega x \\
\frac{dz}{dt} &= 0.
\end{aligned}$$

The last of these equations says that $z = z_0$, a constant. The first two can be combined to give

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0,$$

which just says

$$\frac{d}{dt}(x^2 + y^2) = 0.$$

So, $x^2 + y^2 = a^2$, where a is a constant. So the integral curves are concentric circles about the z-axis in planes parallel to the $x \sim y$ plane.

To end this section we note, that to some extent, the factor λ is redundant. This is because we can always remove it by re-parametrising the curves. To do this we define a function τ of t by

$$\tau = \int \lambda(t) dt.$$

So that $d\tau/dt = \lambda(t)$ and the equations for the integral curves can be written as

$$\frac{d\boldsymbol{r}}{d\tau} = \boldsymbol{F}(\boldsymbol{r}(\tau)).$$

So, under most circumstances, we can set $\lambda = 1$ by a re-parameterisation. However, such a parameterisation may be much more complicated than one with a λ different from 1.

8 Lecture 5

Conservative Vector Fields

The gradient of a differentiable scalar field is a vector field. Can any vector field be written as a gradient? The answer is no, in general. However, the class of vector fields which can be written as a gradient, called *conservative vector fields*, are very important in mathematics and physics.

$\diamond$ Conservative Vector Field: A vector field is said to be conservative in a region $\Omega \subseteq \mathbb{R}^3$ if there exists a scalar field ψ, differentiable on Ω, such that

$$\boldsymbol{F} = \boldsymbol{\nabla}\psi.$$

The scalar field ψ, with $\boldsymbol{F} = \boldsymbol{\nabla}\psi$, is known as the *potential* for the conservative vector field $\boldsymbol{F}$.

Given a vector field $\boldsymbol{F}$ how do we know if it is conservative or not? Mathematically, we need to know the necessary and sufficient conditions for $\boldsymbol{F}$ to be a conservative vector field. If $\boldsymbol{F}$ *is* conservative, $\boldsymbol{F} = \boldsymbol{\nabla}\boldsymbol{\psi}$, then from identity 8 (Lecture 3) we see that $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$. So the condition $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$ is a necessary condition for $\boldsymbol{F}$ to be conservative. In fact, $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$ is a necessary *and* sufficient condition for $\boldsymbol{F}$ to be conservative.

$$\diamond \ \boldsymbol{F} \text{ a conservative vector field} \Leftrightarrow \boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}.$$

This statement is most easily proved as a corollary to what is known as Stokes' theorem, which we will look at in Lecture 12.

$\blacklozenge$**Example** Show that $\boldsymbol{r}$ is a conservative vector field.

Solution: From Lecture 2 we know that $\boldsymbol{\nabla}\times\boldsymbol{r} = \boldsymbol{0}$, so $\boldsymbol{r}$ is certainly conservative. In fact, from the first example of Lecture 2 we have

$$\boldsymbol{\nabla} r^n = n r^{n-2}\boldsymbol{r}.$$

So for $n = 2$ we get $\boldsymbol{\nabla} r^2 = 2\boldsymbol{r}$ or $\boldsymbol{r} = \boldsymbol{\nabla}(r^2/2)$. So $\boldsymbol{r}$ is a conservative vector field with *potential* $r^2/2$.

In physics conservative vector fields represent conservative forces (gravitational and electrostatic forces, for example). For these conservative forces the potential is just the potential energy associated with the force. We need to be able to find the potential for any conservative field.

$\diamond$ Finding Potentials For Conservative Fields: Firstly, we observe that the potential cannot be unique, $\boldsymbol{\nabla}\psi = \boldsymbol{\nabla}(\psi + c)$, where c is any constant. However, we can say that the potential is unique modulo an additive constant. Suppose we are given a conservative vector field $\boldsymbol{F}$, then we know that

$$\boldsymbol{F} = \boldsymbol{\nabla}\psi,$$

for some scalar field ψ. For $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$ we have the three scalar equations

$$\begin{aligned}\frac{\partial\psi}{\partial x} &= F_1\\ \frac{\partial\psi}{\partial y} &= F_2\\ \frac{\partial\psi}{\partial z} &= F_3.\end{aligned}$$

These three coupled, first order, partial differential equations are quite easily integrated to find ψ. Generally, we proceed as follows

- Choose one of the three equations (the easiest) and integrate ("partially", i.e. holding the other two variables constant). Suppose we integrate the first equation with respect to x:

  $$\psi = \int F_1 dx + f(y, z).$$

  Notice the, as yet arbitrary, function $f(y, z)$. This is the "constant" of integration, when we integrate the partial derivative with respect to x we must get an arbitrary function of the other two variables y and z which were held constant. Remember, when you differentiate partially you hold the other variables constant, when you perform the inverse operation (anti-differentiation or integration) you must also hold the other variables constant – any constant of integration can then involve an arbitrary function of these other variables. The x dependence of ψ has now been determined.

- Now take our expression $\psi = \int F_1 dx + f(y, z)$ and substitute it into the next equation, we have

  $$\begin{aligned}\frac{\partial\psi}{\partial y} &= \frac{\partial}{\partial y}\int F_1 dx + \frac{\partial f}{\partial y}(y, z)\\ &= F_2\\ \text{or,}\quad \frac{\partial f}{\partial y}(y, z) &= F_2 - \int \frac{\partial F_1}{\partial y}\, dx.\end{aligned}$$

  This last equation is a partial differential equation for f in two independent variables y and z. We proceed as before and integrate partially, this time with respect to y.

$$f(y,z) = \int F_2 dy - \int \left(\int \frac{\partial F_1}{\partial y} dx \right) dy + g(z)$$

Notice the, as yet, arbitrary function of z, $g(z)$. The "constant" of integration must be a function of the other variables, in this case only z remains. We now have

$$\psi = \int F_1 dx + f(y,z)$$

where f is given as above, in terms of $g = g(z)$. So only the z dependence of ψ remains to be determined.

- The function g can now be determined by substituting our expression for ψ into the last equation. After integration with respect to z we will have determined g and hence ψ up to an arbitrary constant.

The procedure we have outlined may seem a little cumbersome in the full generality in which we have described it here, but in practice it is quite easy to implement.

♦**Example** The vector field $\boldsymbol{G} = yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}$ is conservative (check it! Show that $\boldsymbol{\nabla} \times \boldsymbol{G} = \boldsymbol{0}$), find a potential for $\boldsymbol{G}$.

Solution: $\boldsymbol{G}$ is conservative so there exists a potential ψ, say, such that $\boldsymbol{G} = \boldsymbol{\nabla}\psi$. Our three component equations are

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= yz \\ \frac{\partial \psi}{\partial y} &= xz \\ \frac{\partial \psi}{\partial z} &= xy. \end{aligned}$$

Integrate the first equation, with respect to x, to get

$$\psi = \int yz\, dx + f(y,z) = xyz + f(y,z).$$

Now substitute this into the next equation

$$\begin{aligned} \frac{\partial \psi}{\partial y} &= xz + \frac{\partial f}{\partial y}(y,z) \\ &= xz \\ \text{so, } \frac{\partial f}{\partial y}(y,z) &= 0. \end{aligned}$$

Hence, $f = g(z)$, a function of z only. So we have, thus far, $\psi = xyz + g(z)$. We now substitute this into the last of our equations

$$\begin{aligned} \frac{\partial \psi}{\partial z} &= xy + g'(z) \\ &= xy \\ \text{or, } g'(z) &= 0. \end{aligned}$$

So, g is constant and our potential is $\psi = xyz$. We don't worry about the constant, ψ is only determined up to an additive constant.

♦**Example** Find the potential for the conservative vector field $\boldsymbol{F} = 2x\boldsymbol{i} + (2y + z + 1)\boldsymbol{j} + (y + 2z)\boldsymbol{k}$.

Solution: You should first check that $\boldsymbol{F}$ is in fact conservative (check its curl). Let the potential be ψ, then our three component equations are

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= 2x \\ \frac{\partial \psi}{\partial y} &= 2y + z + 1 \\ \frac{\partial \psi}{\partial z} &= y + 2z. \end{aligned}$$

Integrating the first equation we get

$$\psi = x^2 + f(y, z),$$

with $f = f(y, z)$ an arbitrary function of (y, z). Next, substitute this into the second equation

$$\begin{aligned} \frac{\partial \psi}{\partial y} &= \frac{\partial f}{\partial y}(y, z) \\ &= 2y + z + 1, \end{aligned}$$

now integrate this equation with respect to y, to get

$$f(y, z) = y^2 + zy + y + g(z),$$

where $g = g(z)$ is an arbitrary function of z. We have, thus far $\psi = x^2 + y^2 + yz + y + g(z)$. Finally, we substitute this expression for ψ into the last equation

$$\begin{aligned} \frac{\partial \psi}{\partial z} &= y + g'(z) \\ &= y + 2z \\ \text{so that } g'(z) &= 2z. \end{aligned}$$

We now integrate the last expression with respect to z, the result is $g = z^2$+constant. We have our potential $\psi = x^2 + y^2 + z^2 + yz + y$, you should always check your potential by computing $\boldsymbol{\nabla}\psi$.

9 Lecture 6

Another Characterisation of Conservative Fields

Those of you familiar with elementary physics will recall that the potential energy due to a mass m in the Earth' s gravitational field, near the Earth's surface is given by mgh, where h is the height above the surface. If I move the mass from a height of 2m to a height of 3m the change in potential energy is mg. It makes no difference at all how I get from height 2m to the height 3m, the potential difference is the same no matter which path I follow. Now the potential energy is some sort of "integral" of its associated conservative force. So we suspect that there should be some type of *path independence* associated to this integral of the conservative force. These intuitive musings are given a precise form in the following theorem.

◇ Theorem: *Let $\boldsymbol{F}$ be a vector field defined on an open, connected neighbourhood $\Omega \subseteq \mathbb{R}^3$. The line integral*

$$\int_{\mathcal{C}} \boldsymbol{F}.d\boldsymbol{r},$$

along a path $\mathcal{C}$ between two points P_0 and P_1 is independent of the path $\mathcal{C}$ iff $\boldsymbol{F}$ is a conservative vector field. If $\boldsymbol{F}$ is conservative with potential ψ then

$$\int_{\mathcal{C}} \boldsymbol{F}.d\boldsymbol{r} = \psi(P_1) - \psi(P_0).$$

Proof: This is an "iff" (i.e. if and only if) theorem, so we must provide two proofs. Firstly we will prove
$\boldsymbol{F}$ conservative ⇒ path independence of the line integral.
Secondly, we will prove
path independence of the line integral ⇒ $\boldsymbol{F}$ conservative.
Assuming $\boldsymbol{F}$ to be conservative we have that $\boldsymbol{F} = \boldsymbol{\nabla}\psi$ for some ψ. Then,

$$\begin{aligned} \boldsymbol{F}.d\boldsymbol{r} &= (\boldsymbol{\nabla}\psi).d\boldsymbol{r} \\ &= \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial z}dz \\ &= d\psi. \end{aligned}$$

So if $\mathcal{C}$ is *any* piecewise smooth curve from P_0 to P_1 we have

$$\int_{\mathcal{C}} \boldsymbol{F}.d\boldsymbol{r} = \int_{\mathcal{C}} d\psi = \psi(P_1) - \psi(P_0).$$

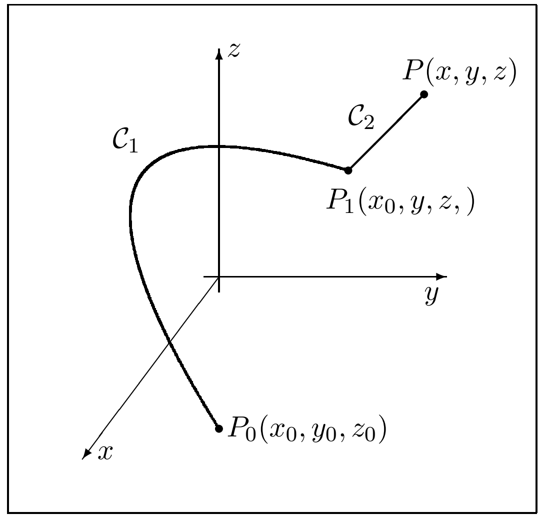
The integral depends only on the end points P_0 and P_1, not on the particular path $\mathcal{C}$. This completes the first half of the proof.

Now we assume that the integral is path independent and prove that it is conservative.

Let $P_0 = (x_0, y_0, z_0)$ be fixed in Ω and let $P = (x, y, z)$ be any point in Ω. The integral

$$\int_{\mathcal{C}} \boldsymbol{F}.d\boldsymbol{r}$$

is well defined and independent of the path, for any piecewise smooth path joining P_0 to P.



So the function

$$\psi(x, y, z) = \int_{\mathcal{C}} \boldsymbol{F}.d\boldsymbol{r}$$

is well defined and depends only on the points $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$. We may choose $\mathcal{C}$ in any way we please with no effect on the function $= \psi$. We take $\mathcal{C}$ to be made up of two pieces, a smooth curve $\mathcal{C}_1$ from P_0 to $P_1 = (x_0, y, z)$ and a straight line segment, $\mathcal{C}_2$ joining P_1 to P. Then

$$\begin{aligned}
\psi = \int_{\mathcal{C}} \boldsymbol{F}.d\boldsymbol{r} &= \int_{\mathcal{C}_1} \boldsymbol{F}.d\boldsymbol{r} + \int_{\mathcal{C}_2} \boldsymbol{F}.d\boldsymbol{r} \\
&= \int_{\mathcal{C}_1, x=x_0} \boldsymbol{F}.d\boldsymbol{r} + \int_{\mathcal{C}_2,\ (y,z)} \boldsymbol{F}.d\boldsymbol{r} \\
&= \int_{\mathcal{C}_1, x=x_0} \boldsymbol{F}.d\boldsymbol{r} + \int_{x_0}^{x} F_1\, dx,
\end{aligned}$$

where we have used the fact that if we are integrating along a curve with (y, z) fixed we have $d\boldsymbol{r} = dx\boldsymbol{i}$ along that curve, so $\boldsymbol{F}.d\boldsymbol{r} = F_1 dx$ on the curve. Now differentiate this expression for ψ with respect to x to find

$$\begin{aligned}
\frac{\partial \psi}{\partial x} &= \frac{\partial}{\partial x}\left(\int_{\mathcal{C}_1, x=x_0} \boldsymbol{F}.d\boldsymbol{r} + \int_{x_0}^{x} F_1\, dx\right) \\
&= \frac{\partial}{\partial x}\left(\int_{x_0}^{x} F_1\, dx\right) \\
&= F_1,
\end{aligned}$$

the differential of the first integral vanishes because $x = x_0$ for that term. Similar arguments will show that

$$\frac{\partial \psi}{\partial y} = F_2 \text{ and } \frac{\partial \psi}{\partial z} = F_3.$$

Consequently, $\boldsymbol{F} = \boldsymbol{\nabla}\psi$, $\boldsymbol{F}$ is a conservative vector field.

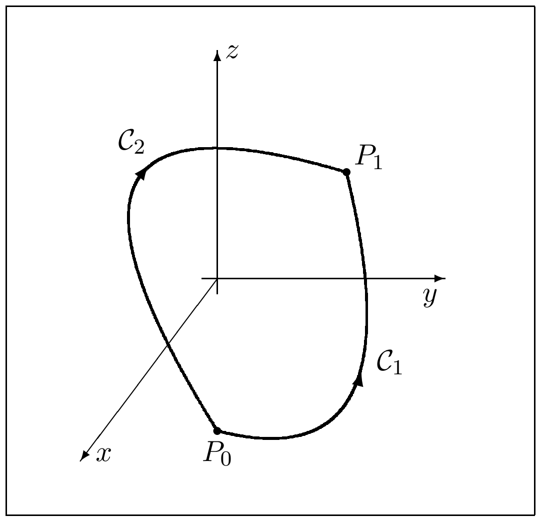
A simple corollary to our theorem is

◇ Corollary: A vector field $\boldsymbol{F}$ is conservative in a region Ω iff the line integral of $\boldsymbol{F}$ around arbitrary closed curves in Ω vanishes.
Proof: Consider a closed curve $\mathcal{C}$.

Pick any two distinct points on $\mathcal{C}$, P_0 and P_1, the curve $\mathcal{C}$ is now divided in two pieces $\mathcal{C}_1$ and $\mathcal{C}_2$ as shown – note the orientations of the curves. We write, "descriptively" $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$. We have

$$\int_{\mathcal{C}} \boldsymbol{F}.d\boldsymbol{r} = \int_{\mathcal{C}_1} \boldsymbol{F}.\, d\boldsymbol{r} - \int_{\mathcal{C}_2} \boldsymbol{F}.\, d\boldsymbol{r}.$$

The two integrals on the right must be path independent.



Depending only on the same two end points P_0 and P_1. Consequently they are equal, the result follows. The "iff" character of the corollary follows from the iff character of our theorem.

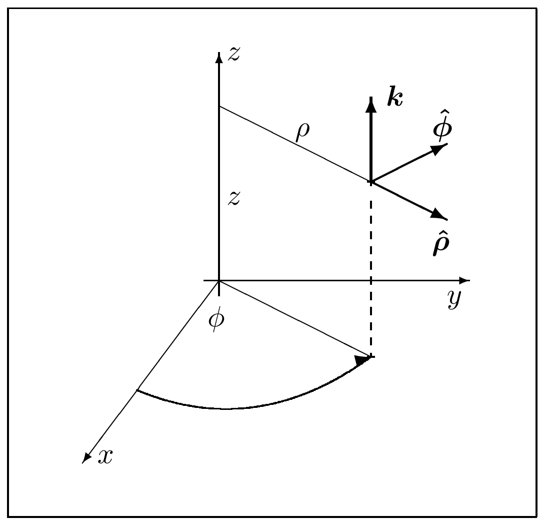
10 Lecture 7

In many applications or problems our rectangular Cartesian coordinates are not well adapted to the geometry or symmetries of the situation. In fact, they can sometimes positively hinder the finding of a solution. In such situations one seeks coordinates adapted to the geometry at hand. In treating circular motion in the plane the use of plane polar coordinates greatly simplifies the mathematics . The central force problem in Newtonian mechanics (e.g. satellite motion about the Sun) is greatly simplified once spherical polar coordinates are introduced. If we are to make any practical use of our vector differential operators we will need to know how to translate them into different coordinate systems. In this course we will treat two systems, *cylindrical polar coordinates* and *spherical polar coordinates*.

Cylindrical Polar Coordinates

A point with Cartesian coordinates (x, y, z) is assigned cylindrical polar coordinates (ρ, ϕ, z) where the two sets of coordinates are related as follows

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z. \end{aligned} \tag{1}$$



At $P = (\rho, \phi, z)$ we have three mutually perpendicular unit vectors – one for each of the coordinate directions (cf. x with $\boldsymbol{i}$, y with $\boldsymbol{j}$ and z with $\boldsymbol{k}$). Each of these unit vectors points in the direction in which the associated coordinate is increasing

$$\begin{array}{lll} \hat{\boldsymbol{\rho}} & \perp & z\text{-axis through } P \\ \hat{\boldsymbol{\phi}} & \perp & \text{the plane containing } z \text{ and the } z\text{-axis} \\ \boldsymbol{k} & & \text{as for Cartesian coordinates.} \end{array}$$

The position vector $\boldsymbol{r}$ is

$$\begin{aligned} \boldsymbol{r} &= x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \\ &= \rho\hat{\boldsymbol{\rho}} + z\boldsymbol{k}. \end{aligned}$$

As a consequence of the above definitions we have

$$\begin{aligned} \hat{\boldsymbol{\rho}} &= \cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}. \end{aligned} \tag{2}$$

These expressions can be inverted to give

$$\begin{aligned} \boldsymbol{i} &= \cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \boldsymbol{j} &= \sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\,\hat{\boldsymbol{\phi}}. \end{aligned} \tag{3}$$

Our objective now is find expressions for the gradient, divergence, curl and Laplacian in terms of the cylindrical coordinates and the associated unit vectors. The essentially new feature which arises here is that the coordinate unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ depend on the point P under consideration. In fact, we have (from equations (2)) that they are both vector functions of ϕ. Their ϕ derivatives are

$$\begin{aligned} \frac{\partial\hat{\boldsymbol{\rho}}}{\partial\phi} &= \hat{\boldsymbol{\phi}} \\ \frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} &= -\hat{\boldsymbol{\rho}}. \end{aligned}$$

<u>The Gradient</u>

Let ψ be a C^1 scalar field, applying the chain rule we get, using (1) above

$$\begin{aligned} \frac{\partial\psi}{\partial\rho} &= \frac{\partial x}{\partial\rho}\frac{\partial\psi}{\partial x} + \frac{\partial y}{\partial\rho}\frac{\partial\psi}{\partial y} + \frac{\partial z}{\partial\rho}\frac{\partial\psi}{\partial z} \\ &= \cos\phi\,\frac{\partial\psi}{\partial x} + \sin\phi\,\frac{\partial\psi}{\partial y} \\ \frac{\partial\psi}{\partial\phi} &= \frac{\partial x}{\partial\phi}\frac{\partial\psi}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial\psi}{\partial y} + \frac{\partial z}{\partial\phi}\frac{\partial\psi}{\partial z} \\ &= -\rho\sin\phi\,\frac{\partial\psi}{\partial x} + \rho\cos\phi\,\frac{\partial\psi}{\partial y}. \end{aligned}$$

Then, by first using equations (3)

$$\begin{aligned} \boldsymbol{\nabla}\psi &= \frac{\partial\psi}{\partial x}\boldsymbol{i} + \frac{\partial\psi}{\partial y}\boldsymbol{j} + \frac{\partial\psi}{\partial z}\boldsymbol{k} \\ &= \frac{\partial\psi}{\partial x}(\cos\phi\,\hat{\boldsymbol{\rho}} - \sin\phi\,\hat{\boldsymbol{\phi}}) + \frac{\partial\psi}{\partial y}(\sin\phi\,\hat{\boldsymbol{\rho}} + \cos\phi\,\hat{\boldsymbol{\phi}}) + \frac{\partial\psi}{\partial z}\boldsymbol{k} \\ &= \left(\cos\phi\,\frac{\partial\psi}{\partial x} + \sin\phi\,\frac{\partial\psi}{\partial y}\right)\hat{\boldsymbol{\rho}} + \left(-\sin\phi\,\frac{\partial\psi}{\partial x} + \cos\phi\,\frac{\partial\psi}{\partial y}\right)\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\boldsymbol{k}. \end{aligned}$$

Next using our chain rule expressions for the ρ and ϕ derivatives

$$\diamond \quad \boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\boldsymbol{k}.$$

The Divergence

Let $\boldsymbol{F}$ be a vector field, in cylindrical polar coordinates we have

$$\boldsymbol{F} = F_\rho\,\hat{\boldsymbol{\rho}} + F_\phi\,\hat{\boldsymbol{\phi}} + F_z\,\boldsymbol{k},$$

where F_ρ, F_ϕ and F_z are the components of $\boldsymbol{F}$ with respect to the three unit cylindrical coordinate vectors. Now take the divergence of this expression

$$\begin{aligned} \boldsymbol{\nabla}\boldsymbol{.}\boldsymbol{F} &= \boldsymbol{\nabla}\boldsymbol{.}(F_\rho\,\hat{\boldsymbol{\rho}}) + \boldsymbol{\nabla}\boldsymbol{.}(F_\phi\,\hat{\boldsymbol{\phi}}) + \boldsymbol{\nabla}\boldsymbol{.}(F_z\,\boldsymbol{k}) \\ &= F_\rho\,\boldsymbol{\nabla}\boldsymbol{.}\hat{\boldsymbol{\rho}} + (\hat{\boldsymbol{\rho}}\boldsymbol{.}\boldsymbol{\nabla})F_\rho + F_\phi\,\boldsymbol{\nabla}\boldsymbol{.}\hat{\boldsymbol{\phi}} + (\hat{\boldsymbol{\phi}}\boldsymbol{.}\boldsymbol{\nabla})F_\phi + \boldsymbol{k}\boldsymbol{.}\boldsymbol{\nabla}F_z. \end{aligned}$$

In the second line we have made use of identity (2) of Lecture 3 and the fact that $\boldsymbol{k}$ is a constant vector. To complete our calculation of the divergence of $\boldsymbol{F}$ we need to calculate the divergence of each of the two unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$. Using the definition and the equations (2) we have

$$\begin{aligned} \boldsymbol{\nabla}\boldsymbol{.}\hat{\boldsymbol{\rho}} &= -\sin\phi\,\frac{\partial\phi}{\partial x} + \cos\phi\,\frac{\partial\phi}{\partial y} \\ &= -\sin\phi\left(-\frac{\sin\phi}{\rho}\right) + \cos\phi\left(\frac{\cos\phi}{\rho}\right) \\ &= \frac{1}{\rho} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{\nabla}\boldsymbol{.}\hat{\boldsymbol{\phi}} &= -\cos\phi\,\frac{\partial\phi}{\partial x} - \sin\phi\,\frac{\partial\phi}{\partial y} \\ &= -\cos\phi\left(-\frac{\sin\phi}{\rho}\right) - \sin\phi\left(\frac{\cos\phi}{\rho}\right) \\ &= 0. \end{aligned}$$

Finally, we note that

$$\hat{\boldsymbol{\rho}}\boldsymbol{.}\boldsymbol{\nabla} = \frac{\partial}{\partial\rho}, \quad \hat{\boldsymbol{\phi}}\boldsymbol{.}\boldsymbol{\nabla} = \frac{1}{\rho}\frac{\partial}{\partial\phi}, \text{ and } \boldsymbol{k}\boldsymbol{.}\boldsymbol{\nabla} = \frac{\partial}{\partial z}.$$

Putting all this together,

$$\diamond \quad \boldsymbol{\nabla}\boldsymbol{.}\boldsymbol{F} = \frac{F_\rho}{\rho} + \frac{\partial F_\rho}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z}.$$

♦**Example** Find the divergence of $\boldsymbol{F} = \rho^2\,\hat{\boldsymbol{\rho}} - \rho\cos\phi\,\hat{\boldsymbol{\phi}}$.

Solution: A direct application of the definition gives

$$\begin{aligned}
\boldsymbol{\nabla}.\boldsymbol{F} &= \frac{\rho^2}{\rho} + \frac{\partial(\rho^2)}{\partial\rho} + \frac{1}{\rho}\frac{\partial(-\rho\cos\phi)}{\partial\phi} + \frac{\partial(0)}{\partial z} \\
&= \rho + 2\rho + \sin\phi \\
&= 3\rho + \sin\phi.
\end{aligned}$$

The Curl

The curl of a vector function $\boldsymbol{F} = F_\rho\,\hat{\boldsymbol{\rho}} + F_\phi\,\hat{\boldsymbol{\phi}} + F_z\,\boldsymbol{k}$ can be calculated using the same type of technique as was used for the divergence. We simply use identity 3 (Lecture 3), rather than identity 2. You might like to try this as an exercise. The result of the calculation is

$$\diamond \quad \boldsymbol{\nabla}\times\boldsymbol{F} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \boldsymbol{k} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}.$$

♦**Example** Find the curl of $\boldsymbol{G} = \rho\,\hat{\boldsymbol{\rho}} + \rho\sin\phi\,\hat{\boldsymbol{\phi}} + \rho\,\boldsymbol{k}$.

Solution: Substituting for the components of $\boldsymbol{G}$ into the definition we have

$$\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{G} &= \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \boldsymbol{k} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \rho & \rho(\rho\sin\phi) & \rho \end{vmatrix} \\
&= \frac{1}{\rho}\left\{\left[\frac{\partial(\rho)}{\partial\phi} - \frac{\partial(\rho^2\sin\phi)}{\partial z}\right]\hat{\boldsymbol{\rho}} - \left[\frac{\partial(\rho)}{\partial\rho} - \frac{\partial(\rho)}{\partial z}\right]\rho\,\hat{\boldsymbol{\phi}} + \left[\frac{\partial(\rho^2\sin\phi)}{\partial\rho} - \frac{\partial(\rho)}{\partial\phi}\right]\boldsymbol{k}\right\} \\
&= -\hat{\boldsymbol{\phi}} + 2\sin\phi\,\boldsymbol{k}.
\end{aligned}$$

The Laplacian

The Laplacian of a scalar field ψ is calculated as $\boldsymbol{\nabla}.(\boldsymbol{\nabla}\psi)$, the result is

$$\diamond \quad \nabla^2\psi = \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}.$$

To calculate the Laplacian of a vector field we *must* use identity 9 (Lecture 3) to consistently take account of the various derivatives of our unit vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$. That is we calculate the Laplacian of $\boldsymbol{F}$ as

$$\nabla^2\boldsymbol{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{F}).$$

♦**Example** Find the Laplacian of $\boldsymbol{F} = \hat{\boldsymbol{\rho}} - \rho\,\hat{\boldsymbol{\phi}} + \cos\phi\,\boldsymbol{k}$

<u>Solution</u>: We must first calculate the curl and divergence of $\boldsymbol{F}$, the results of these two calculations are

$$\begin{aligned} \boldsymbol{\nabla.F} &= \frac{1}{\rho} \\ \boldsymbol{\nabla} \times \boldsymbol{F} &= -\frac{\sin\phi}{\rho}\,\hat{\boldsymbol{\rho}} - 2\boldsymbol{k}. \end{aligned}$$

We can now calculate the gradient of the first expression (this will give $\boldsymbol{\nabla}(\boldsymbol{\nabla.F})$) and the curl of the second expression (to give the curl(curl$\boldsymbol{F}$)). The results are

$$\begin{aligned} \boldsymbol{\nabla}(\boldsymbol{\nabla.F}) &= -\frac{1}{\rho^2}\,\hat{\boldsymbol{\rho}} \\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla} \times \boldsymbol{F}) &= \frac{\cos\phi}{\rho^2}\boldsymbol{k}. \end{aligned}$$

Finally, we combine these expressions to give the Laplacian

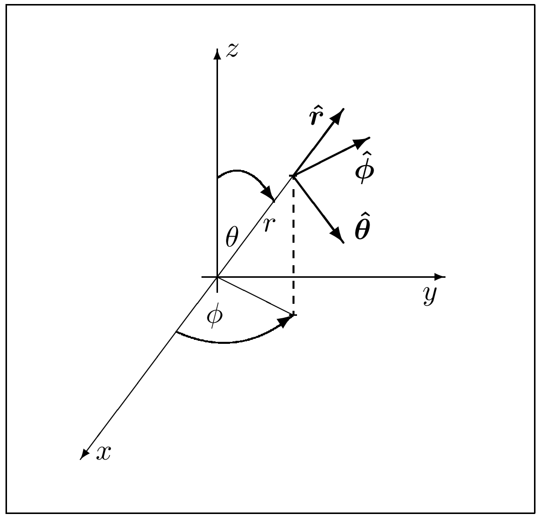
$$\nabla^2\boldsymbol{F} = -\frac{1}{\rho^2}\,\hat{\boldsymbol{\rho}} - \frac{\cos\phi}{\rho^2}\boldsymbol{k}.$$

11 Lecture 8

Spherical Polar Coordinates

A point with Cartesian coordinates (x, y, z) is assigned spherical polar coordinates (r, θ, ϕ) where the two sets of coordinates are related as follows

$$\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta. \end{aligned}$$



At $P = (r, \theta, \phi)$ we have three mutually perpendicular unit vectors – one for each of the coordinate directions,

$$\begin{aligned} \hat{\boldsymbol{r}} &= \cos\phi\sin\theta\,\boldsymbol{i} + \sin\phi\sin\theta\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k} \\ \hat{\boldsymbol{\theta}} &= \cos\phi\cos\theta\,\boldsymbol{i} + \sin\phi\cos\theta\,\boldsymbol{j} - \sin\theta\,\boldsymbol{k} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}. \end{aligned}$$

The position vector $\boldsymbol{r}$ is

$$\begin{aligned} \boldsymbol{r} &= r\,(\cos\phi\sin\theta\,\boldsymbol{i} + \sin\phi\sin\theta\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}) \\ &= r\hat{\boldsymbol{r}}. \end{aligned}$$

The same methods we used to calculate grad, div and curl in the last lecture can be applied here. We will not go through the details, as the computations are messy. The concepts, however, are reasonably straightforward.

<u>The Gradient</u>

$$\diamond \quad \boldsymbol{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}.$$

The Divergence

For a differentiable vector field $\boldsymbol{F} = F_r\,\hat{\boldsymbol{r}} + F_\theta\,\hat{\boldsymbol{\theta}} + F_\phi\,\hat{\boldsymbol{\phi}}$ we have

$$\diamond \quad \boldsymbol{\nabla}.\boldsymbol{F} = \frac{2}{r}F_r + \frac{\partial F_r}{\partial r} + \frac{\cot\theta}{r}F_\theta + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}.$$

The Curl

$$\diamond \quad \boldsymbol{\nabla}\times\boldsymbol{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\,\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta\,F_\phi \end{vmatrix}.$$

The Laplacian

The Laplacian of a scalar field is calculated as $\boldsymbol{\nabla}.(\boldsymbol{\nabla}\psi)$, the result is

$$\nabla^2\psi = \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}.$$

To find the Laplacian of a vector field we *must* again resort to identity 9.

♦**Example** Show that the vector field $\boldsymbol{A} = \frac{\cot\theta}{r}\,\hat{\boldsymbol{\phi}}$ satisfies Laplace's equation.

Solution: We must show that

$$\nabla^2\boldsymbol{A} \equiv \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{A}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) = \boldsymbol{0}.$$

We first calculate the divergence and curl of $\boldsymbol{A}$, the results are

$$\begin{aligned} \boldsymbol{\nabla}.\boldsymbol{A} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\cot\theta}{r}\right) = 0 \\ \boldsymbol{\nabla}\times\boldsymbol{A} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\,\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r\sin\theta(\cot\theta/r) \end{vmatrix} = -\frac{\hat{\boldsymbol{r}}}{r^2}. \end{aligned}$$

One note of caution here: don't forget the multiplicative factors (r and $r\sin\theta$) in the bottom line of the determinant for curl – a common mistake! Now back to the calculation. We need grad and curl, respectively, of the above two quantities. The first is easy, grad(0) = 0! For the second we have

$$\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\boldsymbol{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ -\frac{1}{r^2} & 0 & 0 \end{vmatrix} = 0$$

and we conclude that $\nabla^2 \boldsymbol{A} = \boldsymbol{0}$.

♦**Example** Show that $f = \cos\theta / r^2$ is a harmonic function.

Solution: Using the expression for the Laplacian of a scalar in spherical polar coordinates we have

$$\begin{aligned}
\nabla^2 f &= \frac{2}{r}\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r^2}\right) + \frac{\partial^2}{\partial r^2}\left(\frac{\cos\theta}{r^2}\right) + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r^2}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left(\frac{\cos\theta}{r^2}\right) \\
&= -4\frac{\cos\theta}{r^4} + 6\frac{\cos\theta}{r^4} - \frac{\cos\theta}{r^4} - \frac{\cos\theta}{r^4} \\
&= 0.
\end{aligned}$$

♦**Example** Find a potential for the conservative vector field

$$\boldsymbol{F} = 2r\cos\theta\,\hat{\boldsymbol{r}} + \frac{1}{r}(\sin\phi\,\cos\theta - r^2\sin\theta)\hat{\boldsymbol{\theta}} + \frac{\cos\phi}{r}\hat{\boldsymbol{\phi}}$$

Solution: In general, if $\boldsymbol{F}$ is a conservative vector field and ψ the associated potential then we have $\boldsymbol{F} = \boldsymbol{\nabla}\psi$. Written out in component form, in spherical polar coordinates this is

$$\begin{aligned}
\frac{\partial\psi}{\partial r} &= F_r \\
\frac{1}{r}\frac{\partial\psi}{\partial\theta} &= F_\theta \\
\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} &= F_\phi.
\end{aligned}$$

The important thing to note here is the appearance of the factors in front of the derivatives on the left side of the equations. In the present example we get for the component equations, after multiplying the factors on the left to the right sides of the equations

$$\begin{aligned}
\frac{\partial\psi}{\partial r} &= 2r\cos\theta \\
\frac{\partial\psi}{\partial\theta} &= \sin\phi\cos\theta - r^2\sin\theta \\
\frac{\partial\psi}{\partial\phi} &= \sin\theta\cos\phi.
\end{aligned}$$

We now proceed in the usual way. Integrate the first equation (partially) to get

$$\psi = r^2\cos\theta + f(\theta, \phi).$$

Now put this into the second equation to get

$$\frac{\partial f}{\partial\theta}(\theta, \phi) = \sin\phi\,\cos\theta.$$

Integrate this expression,

$$f(\theta, \phi) = \sin\phi \sin\theta + g(\phi).$$

We have thus far,

$$\psi = r^2 \cos\theta + \sin\phi \sin\theta + g(\phi).$$

We now substitute this into the last of our equations to get $g'(\phi) = 0$, so g is constant. We have as a potential for $\boldsymbol{F}$

$$\psi = r^2 \cos\theta + \sin\phi \sin\theta.$$

II. THE INTEGRAL THEOREMS

12 Lecture 9

Introduction

This section of the course is devoted to the study of two remarkable theorems, Gauss' theorem and Stokes' theorem. These theorems tie together integration, vector differential operators and topology. The theorems, their generalisations and underlying ideas have had a profound effect on major parts of modern mathematics and mathematical physics.

Before we state and prove our two theorems we will briefly revise some of the theory of integration for curves, surfaces and volumes – our emphasis here will be on calculating such integrals.

Line Integrals

Given a differentiable, parametric curve, $\mathcal{C}$,

$$\begin{aligned} x &= x(t) \\ y &= y(t) \\ z &= z(t), \end{aligned}$$

and a vector field $\boldsymbol{F}$, defined in a neighbourhood of $\mathcal{C}$. We define the *line integral* of $\boldsymbol{F}$ along $\mathcal{C}$ between $P_0 = (x(t_0), y(t_0), z(t_0))$ and $P_1 = (x(t_1), y(t_1), z(t_1))$ as

$$\int_{\mathcal{C}} \boldsymbol{F}.d\boldsymbol{r} = \int_{t=t_0}^{t_1} \boldsymbol{F}(\boldsymbol{r}(t)).\frac{d\boldsymbol{r}}{dt}\, dt$$

♦**Example** Calculate the line integral of $\boldsymbol{F} = -y\boldsymbol{i} + x\boldsymbol{j} + \boldsymbol{k}$ along the curve

$$\begin{aligned} x &= \cos(\pi t) \\ y &= \sin(\pi t) \\ z &= t, \end{aligned}$$

between the points $(1, 0, 0)$ and $(-1, 0, 1)$.

Solution: On the curve we have

$$\begin{aligned}
\boldsymbol{F} &= -\sin(\pi t)\,\boldsymbol{i} + \cos(\pi t)\,\boldsymbol{j} + \boldsymbol{k} \\
\boldsymbol{r} &= \cos(\pi t)\,\boldsymbol{i} + \sin(\pi t)\,\boldsymbol{j} + t\boldsymbol{k} \\
\frac{d\boldsymbol{r}}{dt} &= -\pi\sin(\pi t)\,\boldsymbol{i} + \pi\cos(\pi t)\,\boldsymbol{j} + \boldsymbol{k}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\mathcal{C}(1,0,0)}^{\mathcal{C}(-1,0,1)} \boldsymbol{F}.d\boldsymbol{r} &= \int_{t=0}^{1} \left[\pi\left(\sin^2(\pi t) + \cos^2(\pi t)\right) + 1\right]\,dt \\
&= \int_{t=0}^{1} (1+\pi)\,dt \\
&= 1+\pi.
\end{aligned}$$

Surface Integrals

A differentiable *parametric surface* is a surface the points of which are given parametrically,

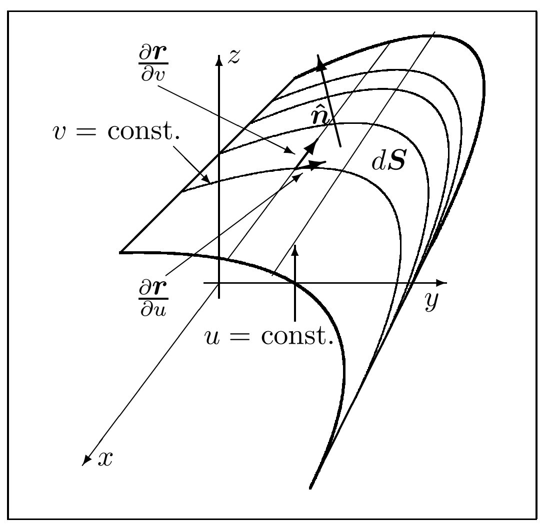
$$\begin{aligned}
x &= x(u,v) \\
y &= y(u,v) \\
z &= z(u,v)
\end{aligned}$$

where u and v are the surface parameters. The x, y and z are differentiable functions of (u,v).

The *oriented surface element*, $d\boldsymbol{S}$, of the parametric surface $\mathcal{S}$ is given by

$$\begin{aligned}
d\boldsymbol{S} &= \hat{\boldsymbol{n}}dS \\
&= \frac{\partial\boldsymbol{r}}{\partial u}\times\frac{\partial\boldsymbol{r}}{\partial v}du\,dv
\end{aligned}$$

Note that $\partial\boldsymbol{r}/\partial u$ is tangent to the curves v = constant and $\partial\boldsymbol{r}/\partial v$ is tangent to the curves u = constant.



This means that their cross product is normal to the surface. This explains its appearance in our expression for $d\boldsymbol{S}$.

The surface area of our surface $\mathcal{S}$ is

$$\int\int_{\mathcal{S}} dS = \int\int_{\mathcal{S}} \left| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right| du\, dv.$$

The *flux* of an integrable vector field $\boldsymbol{F}$ across a surface $\mathcal{S}$ is

$$\int\int_{\mathcal{S}} \boldsymbol{F}.d\boldsymbol{S}.$$

The flux of a vector across a surface has a straightforward interpretation in many circumstances. For example, if $\boldsymbol{v}$ is the velocity vector of a fluid then the flux of $\boldsymbol{v}$ across $\mathcal{S}$ gives the rate of flow of fluid volume across $\mathcal{S}$.

♦**Example** Calculate the flux of $\boldsymbol{r}$ out of the curved portion of the right circular cylinder $x^2 + y^2 = 1, \quad 0 \leq z \leq 1$.

Solution: The natural coordinates here are clearly cylindrical polars. In these coordinates it is then easy to identify the surface parameters. In fact, the surface is just $\rho = 1$ with $0 \leq z \leq 1$. So our surface parameters are ϕ and z, these become the u and v above. With $\rho = 1$ the (parametric) surface equations are

$$\begin{aligned} x &= \cos\phi \\ y &= \sin\phi \\ z &= z. \end{aligned}$$

On the surface, remember we are integrating *on* the surface so we only need the surface values of anything appearing in the integrand. We have

$$\begin{aligned} \boldsymbol{r} &= \cos\phi\, \boldsymbol{i} + \sin\phi\, \boldsymbol{j} + z\boldsymbol{k} \\ \frac{\partial \boldsymbol{r}}{\partial \phi} &= -\sin\phi\, \boldsymbol{i} + \cos\phi\, \boldsymbol{j} \\ \frac{\partial \boldsymbol{r}}{\partial z} &= \boldsymbol{k}. \end{aligned}$$

We can now calculate the various ingredients in the integrand,

$$\begin{aligned} \frac{\partial \boldsymbol{r}}{\partial \phi} \times \frac{\partial \boldsymbol{r}}{\partial z} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \cos\phi\, \boldsymbol{i} + \sin\phi\, \boldsymbol{j}. \end{aligned}$$

Note that this points out of the cylinder, in fact its just $\hat{\boldsymbol{\rho}}$, we were specifically asked for the flux *out* the surface. We now have our oriented surface element,

$$d\boldsymbol{S} = (\cos\phi\, \boldsymbol{i} + \sin\phi\, \boldsymbol{j}) d\phi\, dz.$$

The integrand for the flux integral is

$$\boldsymbol{r}.d\boldsymbol{S} = (\cos^2\phi + \sin^2\phi)d\phi\, dz.$$

So that,

$$\text{Flux of} \quad \boldsymbol{F} = \int_{z=0}^{1}\int_{\phi=0}^{2\pi} d\phi\, dz = 2\pi.$$

Volume Integrals

To integrate or "sum" an integrable quantity ψ over a three dimensional region Ω of $\mathbb{R}^3$ we perform the "triple integral"

$$\int\int\int_{\Omega} \psi\, dV$$

where the volume element is given as $dV = dx\, dy\, dz$ in Cartesian coordinates. For other coordinate systems one needs to calculate the Jacobian. If (u, v, w) is a set of well-defined coordinates then the Jacobian determinant is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

The volume element in this new coordinate system is

$$dV = \frac{\partial(x,y,z)}{\partial(u,v,w)} du\, dv\, dw.$$

For the two coordinate systems discussed in these lecture notes we have

$$\text{Cylindrical Polars} \quad dV = \rho\, d\rho\, d\phi\, dz.$$
$$\text{Spherical Polars} \quad dV = r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

13 Lecture 10

Gauss' Divergence Theorem

◇ Theorem

Let Ω be a bounded, open three dimensional domain in $\mathbb{R}^3$. Assume that Ω lies entirely on the inside of its boundary, $\partial\Omega$, a closed piecewise smooth surface, with outward pointing unit normal $\hat{\boldsymbol{n}}$. If $\boldsymbol{F}$ is a C^1 vector field defined on Ω then,

$$\int\!\!\int\!\!\int_{\Omega} \boldsymbol{\nabla}.\boldsymbol{F}\, dV = \int\!\!\int_{\partial\Omega} \boldsymbol{F}.d\boldsymbol{S},$$

where $d\boldsymbol{S} = \hat{\boldsymbol{n}}\, dS$.

Remarks: This is Gauss' theorem.

- The theorem tells us that the "sum" of all the interior values of div$\boldsymbol{F}$ is equal to the "sum" of the values of $\boldsymbol{F}$ over the bounding surface.

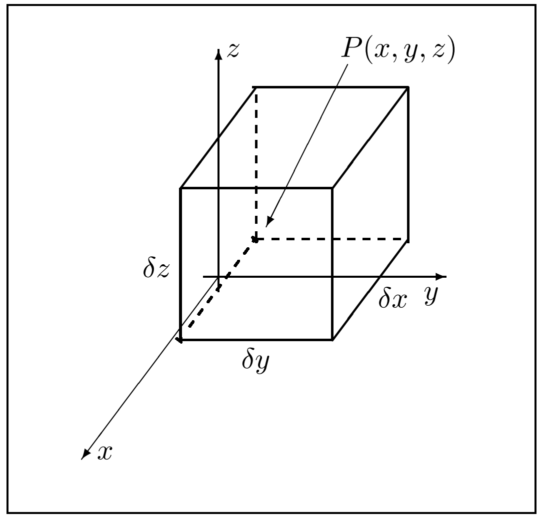
- Note that we require the *outward* normal, the normal pointing from the interior of Ω to the exterior.

- The basic example or picture to keep in mind here is Ω a ball, with boundary $\partial\Omega$ the sphere "skin" of the ball. In fact, in a more abstract setting the most general bounding surface we can have is a sphere with a number of "handles" attached.

- We use the notation $\partial\Omega$ or $\partial\mathcal{S}$ to mean the boundary of the region or surface. The boundary of a three dimensional region is a two dimensional surface. The boundary of a surface will be empty (if the surface is closed, e.g. a sphere) or a curve (a one dimensional space).

Proof: The proof that follows is somewhat heuristic, although with some technical work it can be made quite rigorous. For a thorough proof on a regular domain see [1].

Consider an incremental rectangular prism, with sides δx, δy and δz, in Ω. The prism is located so that $P = (x, y, z)$ is one vertex. For δx, δy and δz sufficiently small the total flux out of the prism in the $\boldsymbol{i}$ direction is



$$\begin{aligned}\delta\Phi_x &\approx F_1(x+\delta x, y, z)\delta y\,\delta z - F_1(x,y,z)\delta y\,\delta z\\ &\approx \frac{\partial F_1}{\partial x}(x,y,z)\,\delta x\,\delta y\,\delta z,\end{aligned}$$

to the lowest order.

Similarly for the fluxes out of the prism in the $\boldsymbol{j}$ and $\boldsymbol{k}$ directions we have

$$\begin{aligned}\delta\Phi_y &\approx \frac{\partial F_2}{\partial y}\delta x\,\delta y\,\delta z\\ \delta\Phi_z &\approx \frac{\partial F_3}{\partial z}\delta x\,\delta y\,\delta z\end{aligned}$$

Writing $\delta V = \delta x\,\delta y\,\delta z$ for the incremental coordinate volume we have for the total flux out of the prism

$$\begin{aligned}\delta\Phi &\approx \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right]\delta V\\ &\approx (\boldsymbol{\nabla . F})\delta V.\end{aligned}$$

Dividing by δV and taking the limit we have

$$\frac{d\Phi}{dV} = \boldsymbol{\nabla . F}.$$

Integrating over the entire region Ω then gives the desired result.

♦**Example** Verify Gauss' theorem when Ω is a sphere of radius a centre the origin and $\boldsymbol{F} = \boldsymbol{r}$.

Solution: We will start with the right hand side, the surface integral. We will need the surface area element for a sphere, this can derived using our earlier techniques.

Clearly the best coordinates for the problem are spherical polars, the bounding surface is simply given as $r = a$. On this surface $\partial\Omega$ we have

$$\begin{aligned} x &= a\sin\theta\,\cos\phi \\ y &= a\sin\theta\,\sin\phi \\ z &= a\cos\theta \end{aligned}$$

This is our parametric form of the bounding spherical surface (our surface parameters are just the θ and ϕ coordinates). Now,

$$\begin{aligned} \frac{\partial \boldsymbol{r}}{\partial\theta} &= a\cos\theta\,\cos\phi\,\boldsymbol{i} + a\cos\theta\,\sin\phi\,\boldsymbol{j} - a\sin\theta\,\boldsymbol{k} \\ \frac{\partial \boldsymbol{r}}{\partial\phi} &= -a\sin\theta\,\sin\phi\,\boldsymbol{i} + a\sin\theta\,cos\phi\,\boldsymbol{j}. \end{aligned}$$

Whence,

$$\begin{aligned} \frac{\partial \boldsymbol{r}}{\partial\theta}\times\frac{\partial \boldsymbol{r}}{\partial\phi} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a\cos\theta\,\cos\phi & a\cos\theta\,\sin\phi & -a\sin\theta \\ -a\sin\theta\,\sin\phi & a\sin\theta\,\cos\phi & 0 \end{vmatrix} \\ &= a^2\sin\theta\,\hat{\boldsymbol{r}} \end{aligned}$$

We check that this gives the outward normal (as required by Gauss' theorem), it does. If we had an inward normal we would reverse the roles in the cross product to obtain the outward normal. So we have our surface area element (a result you have no doubt seen before!)

$$d\boldsymbol{S} = a^2\sin\theta\,\hat{\boldsymbol{r}}\,d\theta\,d\phi$$

On the bounding sphere $\partial\Omega$ we have

$$\boldsymbol{F} = \boldsymbol{r} = a\hat{\boldsymbol{r}}$$

So our integrand is

$$\boldsymbol{F}.d\boldsymbol{S} = a^3\sin\theta\,d\theta\,d\phi$$

So, in the present case, the right hand side of Gauss' theorem becomes

$$\begin{aligned} \int\!\!\int_{\partial\Omega} \boldsymbol{F}.d\boldsymbol{S} &= a^3\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin\theta\,d\theta\,d\phi \\ &= a^3\int_{\phi=0}^{2\pi}[-\cos\theta]_{\theta=0}^{\pi}\,d\phi \\ &= 4\pi a^3. \end{aligned}$$

Now for the left hand side of Gauss' theorem, the volume integral. We found in Lecture 1 that $\boldsymbol{\nabla}.\boldsymbol{r} = 3$, so we have $\boldsymbol{\nabla}.\boldsymbol{F} = 3$ in the present case. Hence,

$$
\begin{aligned}
\int\int\int_{\Omega} (\boldsymbol{\nabla}.\boldsymbol{F})\, dV &= 3\int\int\int_{\Omega} dV \\
&= 3\,\{\text{Volume of} \quad \Omega\} \\
&= 4\pi a^3.
\end{aligned}
$$

So we have established the equality of the left and right sides of Gauss' theorem, in this case.

14 Lecture 11

Stokes' Theorem

◇ Theorem

Let $\mathcal{S}$ be a piecewise smooth, oriented surface in a domain $\Omega \subseteq \mathbb{R}^3$. Let $\mathcal{S}$ have unit normal $\hat{\boldsymbol{n}}$ and a boundary consisting of one or more piecewise smooth closed curves, oriented positively with respect to $\hat{\boldsymbol{n}}$. The collection of bounding curves is the boundary $\partial\mathcal{S}$ of $\mathcal{S}$. If $\boldsymbol{F}$ is a C^1 vector field in Ω then

$$\int_{\partial\mathcal{S}} \boldsymbol{F}\boldsymbol{.}\, d\boldsymbol{r} = \int\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \boldsymbol{F})\boldsymbol{.}\, d\boldsymbol{S},$$

where $d\boldsymbol{S} = \hat{\boldsymbol{n}} dS$ is the oriented surface element of $\mathcal{S}$.

Remarks: This is Stokes' theorem, the theorem made its first appearance in print in the Smith's Prize paper set by G. C. Stokes (Lucasian Professor of Mathematics at Cambridge University) and taken by James Clerk Maxwell in February 1854. The earliest explicit proof appears to be in a letter from Thomson to Stokes dated July 2, 1850.

- The term positive orientation with respect to $\hat{\boldsymbol{n}}$ means positive according to the "right hand thumb" rule: point your thumb in the direction of $\hat{\boldsymbol{n}}$, the direction of your fingers (from base to tips) then gives the positive sense about $\hat{\boldsymbol{n}}$.
- There will be many right hand sides corresponding to the same left hand side. For a fixed bounding curve there can be many surfaces. Consider a rubber membrane stretched across a wire hoop: the hoop is the bounding curve, it's fixed. However, the membrane can be distorted in an infinite variety of ways.
- The surface $\mathcal{S}$ may be bounded by more than one closed curve. The curved portion of a finite cylinder is bounded by two circles, one at each end.

Before giving a proof of the theorem we need to do a little preliminary work in the form of the following lemma.

Lemma: Suppose that x, y and z are given differentiable functions of u and v. Then if $\boldsymbol{F}$ is a C^1 vector field we have

$$(\boldsymbol{\nabla} \times \boldsymbol{F})\boldsymbol{.} \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right) = \frac{\partial \boldsymbol{r}}{\partial v}\boldsymbol{.}\frac{\partial \boldsymbol{F}}{\partial u} - \frac{\partial \boldsymbol{r}}{\partial u}\boldsymbol{.}\frac{\partial \boldsymbol{F}}{\partial v}.$$

<u>Proof of Lemma</u>: First, cyclicly permute the scalar triple product

$$(\boldsymbol{\nabla} \times \boldsymbol{F}) \boldsymbol{.} \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right) = \frac{\partial \boldsymbol{r}}{\partial u} \boldsymbol{.} \left[\frac{\partial \boldsymbol{r}}{\partial v} \times (\boldsymbol{\nabla} \times \boldsymbol{F}) \right].$$

Next, let us examine in detail the term in the square brackets on the right hand side. The $\boldsymbol{i}$ component of the curl gives

$$\frac{\partial \boldsymbol{r}}{\partial v} \times \left(\boldsymbol{i} \times \frac{\partial \boldsymbol{F}}{\partial x} \right) = \left(\frac{\partial \boldsymbol{r}}{\partial v} \boldsymbol{.} \frac{\partial \boldsymbol{F}}{\partial x} \right) \boldsymbol{i} - \left(\frac{\partial \boldsymbol{r}}{\partial v} \boldsymbol{.} \boldsymbol{i} \right) \frac{\partial \boldsymbol{F}}{\partial x}.$$

Here we have used the expansion for the vector triple product,

$$\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c} = (\boldsymbol{a} \boldsymbol{.} \boldsymbol{c}) \boldsymbol{b} - (\boldsymbol{a} \boldsymbol{.} \boldsymbol{b}) \boldsymbol{c}.$$

Taking the scalar product of the previous expression with $\partial \boldsymbol{r} / \partial u$ gives

$$\frac{\partial \boldsymbol{r}}{\partial v} \boldsymbol{.} \left(\frac{\partial x}{\partial u} \frac{\partial \boldsymbol{F}}{\partial x} \right) - \frac{\partial \boldsymbol{r}}{\partial u} \boldsymbol{.} \left(\frac{\partial x}{\partial v} \frac{\partial \boldsymbol{F}}{\partial x} \right).$$

where we have used the fact that

$$\boldsymbol{i} \boldsymbol{.} \frac{\partial \boldsymbol{r}}{\partial u} = \frac{\partial x}{\partial u} \quad \text{and} \quad \boldsymbol{i} \boldsymbol{.} \frac{\partial \boldsymbol{r}}{\partial v} = \frac{\partial x}{\partial v}.$$

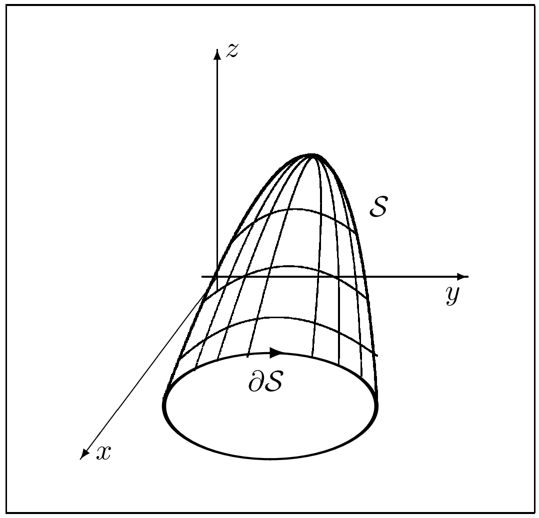
We get similar contributions from the $\boldsymbol{j}$ and $\boldsymbol{k}$ terms. Putting this all together we have

$$\begin{aligned} (\boldsymbol{\nabla} \times \boldsymbol{F}) \boldsymbol{.} \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right) &= \frac{\partial \boldsymbol{r}}{\partial v} \boldsymbol{.} \left(\frac{\partial x}{\partial u} \frac{\partial \boldsymbol{F}}{\partial x} + \frac{\partial y}{\partial u} \frac{\partial \boldsymbol{F}}{\partial y} + \frac{\partial z}{\partial u} \frac{\partial \boldsymbol{F}}{\partial z} \right) - \\ & \quad \frac{\partial \boldsymbol{r}}{\partial u} \boldsymbol{.} \left(\frac{\partial x}{\partial v} \frac{\partial \boldsymbol{F}}{\partial x} + \frac{\partial y}{\partial v} \frac{\partial \boldsymbol{F}}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial \boldsymbol{F}}{\partial z} \right) \\ &= \frac{\partial \boldsymbol{r}}{\partial v} \boldsymbol{.} \frac{\partial \boldsymbol{F}}{\partial u} - \frac{\partial \boldsymbol{r}}{\partial u} \boldsymbol{.} \frac{\partial \boldsymbol{F}}{\partial v}. \end{aligned}$$

Which completes the proof of the lemma. The expression derived in this lemma can be quite economical when calculating the surface integral in Stokes' theorem.

<u>Proof (Stokes' Theorem)</u>: We will give a rather heuristic proof, but as was the case earlier the proof can be made rigorous, see [1].

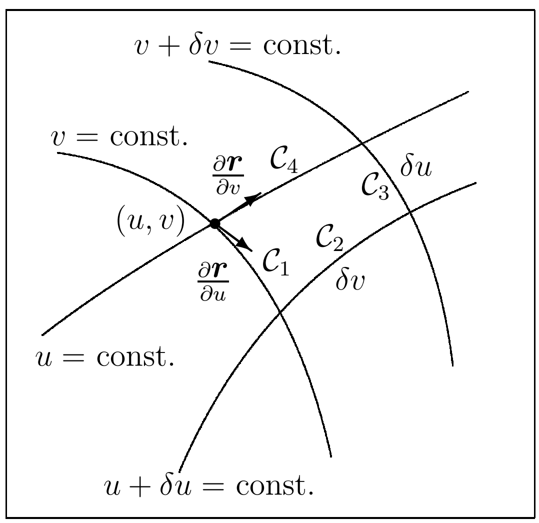
We will also restrict ourselves to the case where $\mathcal{S}$ has just one closed curve, with no self-intersections, as boundary. The extension to the more general case can be easily made.
Let the surface $\mathcal{S}$ be given parametrically with surface parameters u and v. So, making use of our lemma we have



$$\int\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \boldsymbol{F}).d\boldsymbol{S} = \int\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \boldsymbol{F}). \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right) du\, dv$$
$$= \int\int_{\mathcal{S}} \left(\frac{\partial \boldsymbol{r}}{\partial v}.\frac{\partial \boldsymbol{F}}{\partial u} - \frac{\partial \boldsymbol{r}}{\partial u}.\frac{\partial \boldsymbol{F}}{\partial v}\right) du\, dv.$$

We now work with $\int_{\partial\mathcal{S}} \boldsymbol{F}.d\boldsymbol{r}$. This integral is known as the circulation of $\boldsymbol{F}$ about $\partial\mathcal{S}$ and we will denote it by C.

Consider an incremental rhombus inscribed on the surface $\mathcal{S}$ one vertex at the point (u,v) with it sides determined by increments δu and δv along the parameter curves v =constant and u =constant with respective tangent vectors $\partial \boldsymbol{r}/\partial u$ and $\partial \boldsymbol{r}/\partial v$. Let the line segments for the sides of the rhombus be C_1, C_2, C_3 and C_4.



The sides of the rhombus are given by the vectors

$$\frac{\partial \boldsymbol{r}}{\partial u}\delta u \quad \text{and} \quad \frac{\partial \boldsymbol{r}}{\partial v}\delta v.$$

The circulation around the incremental rhombus, δC, is given by

$$\delta C = \int_{C_1} \boldsymbol{F}.d\boldsymbol{r} + \int_{C_2} \boldsymbol{F}.d\boldsymbol{r} + \int_{C_3} \boldsymbol{F}.d\boldsymbol{r} + \int_{C_4} \boldsymbol{F}.d\boldsymbol{r}.$$

We now calculate the four integrals on the right of this equation, to the first order in δu and δv. Firstly, the integral along C_1,

$$\begin{aligned}\int_{C_1} \boldsymbol{F}.d\boldsymbol{r} &= \int_u^{u+\delta u} \boldsymbol{F}.\left(\frac{\partial \boldsymbol{r}}{\partial u}\, du\right) \\ &\approx \boldsymbol{F}(u,v).\left(\frac{\partial \boldsymbol{r}}{\partial u}\right)\delta u,\end{aligned}$$

where we have used the fact that v is constant along C_1 and that the integral can be approximated by the product of the integrand and the length of the interval. In a similar fashion the integral along C_3 can be calculated, noting the direction of integration is opposite the direction given by $\partial \boldsymbol{r}/\partial u$.

$$\int_{C_3} \boldsymbol{F}.d\boldsymbol{r} \approx -\boldsymbol{F}(u, v+\delta v).\left(\frac{\partial \boldsymbol{r}}{\partial u}\right)\delta u.$$

Combining our two integrals for the first and third sides,

$$\begin{aligned}\int_{C_1+C_3} \boldsymbol{F}.d\boldsymbol{r} &\approx -\left[\boldsymbol{F}(u, v+\delta v) - \boldsymbol{F}(u,v)\right].\left(\frac{\partial \boldsymbol{r}}{\partial u}\right)\delta u \\ &\approx -\frac{\partial \boldsymbol{F}}{\partial v}(u,v).\frac{\partial \boldsymbol{r}}{\partial u}\delta u\,\delta v,\end{aligned}$$

to order $\delta u\,\delta v$. Similarly,

$$\int_{C_2+C_4} \boldsymbol{F}.d\boldsymbol{r} \approx \frac{\partial \boldsymbol{F}}{\partial u}(u,v).\frac{\partial \boldsymbol{r}}{\partial v}\delta u\,\delta v.$$

Hence,

$$\delta C \approx \left(\frac{\partial \boldsymbol{F}}{\partial u}.\frac{\partial \boldsymbol{r}}{\partial v} - \frac{\partial \boldsymbol{F}}{\partial v}.\frac{\partial \boldsymbol{r}}{\partial u}\right)\delta u\,\delta v.$$

Denoting by $d\sigma = du\,dv$ the $u \sim v$ area element we have, after taking the limit,

$$\frac{dC}{d\sigma} = \frac{\partial \boldsymbol{F}}{\partial u}.\frac{\partial \boldsymbol{r}}{\partial v} - \frac{\partial \boldsymbol{F}}{\partial v}.\frac{\partial \boldsymbol{r}}{\partial u}.$$

Integrating over $\mathcal{S}$,

$$C(\text{over} \quad \mathcal{S}) = \int\int_{\mathcal{S}} \left(\frac{\partial \boldsymbol{F}}{\partial u}.\frac{\partial \boldsymbol{r}}{\partial v} - \frac{\partial \boldsymbol{F}}{\partial v}.\frac{\partial \boldsymbol{r}}{\partial u}\right) d\sigma.$$

However, all “internal” circulations cancel: counter-clockwise circulations on adjacent rhombi cancel along their common edge (they go in opposite directions there), draw a picture to see this more clearly. Whence,

$$C(\text{over} \quad \mathcal{S}) = C = \int_{\partial \mathcal{S}} \boldsymbol{F}.d\boldsymbol{r}$$

and the theorem is proved.

15 Lecture 12

Stokes' Theorem, continued

In this lecture we continue our look at Stokes' theorem with a simple corollary and an illustrative example.

◇ Corollary

A C^1 vector field is conservative iff $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$.

Proof: We observed back in Lecture 5 that if $\boldsymbol{F}$ was conservative then $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$, identity 8. So we need to prove that *if* $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$ then $\boldsymbol{F}$ is conservative. Suppose $\boldsymbol{F}$ is C^1 in a region Ω of $\mathbb{R}^3$. We will make use of the corollary in Lecture 6, so we start with an arbitrary, piecewise smooth, closed curve $\mathcal{C}$ in the region Ω. Let $\mathcal{S}$ be a piecewise smooth surface bounded by $\mathcal{C}$. Now apply Stokes' theorem,

$$\int_{\mathcal{C}} \boldsymbol{F}.d\boldsymbol{r} = \int\!\!\int_{\mathcal{S}} (\boldsymbol{\nabla} \times \boldsymbol{F}).d\boldsymbol{S} = \boldsymbol{0},$$

since we have assumed $\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$. So by our corollary of Lecture 6 $\boldsymbol{F}$ is conservative.

The following example illustrates the types of calculation needed when dealing with Stokes' theorem.

⧫**Example** Verify Stokes' theorem when $\boldsymbol{F} = y^3\boldsymbol{i} - x^3\boldsymbol{j}$ and $\mathcal{S}$ is the curved surface of the paraboloid of revolution

$$z = x^2 + y^2, \quad 0 \le z \le 4.$$

Solution: We work with the right hand side of Stokes' theorem first. The axial symmetry about the z-axis suggests that the problem will be simplified with the introduction of cylindrical polar coordinates. The surface equation is then, $z = \rho^2$, we can eliminate z in favour of ρ on the surface. This leaves ϕ and ρ as our surface parameters. On $\mathcal{S}$ we have

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= \rho^2 \\ \boldsymbol{r} &= \rho(\cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}) + \rho^2\boldsymbol{k} \end{aligned}$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial \rho} &= \cos\phi\, \boldsymbol{i} + \sin\phi\, \boldsymbol{j} + 2\rho \boldsymbol{k} \\
\frac{\partial \boldsymbol{r}}{\partial \phi} &= -\rho \sin\phi\, \boldsymbol{i} + \rho\cos\phi\, \boldsymbol{j}.
\end{aligned}
$$

So that

$$
\begin{aligned}
\frac{\partial \boldsymbol{r}}{\partial \rho} \times \frac{\partial \boldsymbol{r}}{\partial \phi} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\phi & \sin\phi & 2\rho \\ -\rho\sin\phi & \rho\cos\phi & 0 \end{vmatrix} \\
&= -2\rho^2(\cos\phi\, \boldsymbol{i} + \sin\phi\, \boldsymbol{j}) + \rho \boldsymbol{k} \\
&= -2\rho^2 \hat{\boldsymbol{\rho}} + \rho \boldsymbol{k}.
\end{aligned}
$$

Note that this vector points "upward", i.e. in the positive $\boldsymbol{k}$ sense. This means that the positive sense of integration around the bounding curve, the circle radius 2 in the $z = 4$ plane, will be counter-clockwise about the positive z-axis. We now have our oriented surface element,

$$
d\boldsymbol{S} = \left[-2\rho^2(\cos\phi\, \boldsymbol{i} + \sin\phi\, \boldsymbol{j}) + \rho \boldsymbol{k}\right] d\rho\, d\phi.
$$

Next we need to calculate the curl of $\boldsymbol{F}$,

$$
\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{F} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 & -x^3 & 0 \end{vmatrix} \\
&= -3(x^2 + y^2)\boldsymbol{k} \\
&= -3\rho^2 \boldsymbol{k}.
\end{aligned}
$$

Our integrand for the surface integral is

$$
(\boldsymbol{\nabla} \times \boldsymbol{F}).\, d\boldsymbol{S} = -3\rho^3\, d\rho\, d\phi.
$$

For the limits of integration over the surface we have $0 \le \phi < 2\pi$ and as z has limits $0 \le z < 4$ we see that $\rho = \sqrt{z}$ has the limits $0 \le \rho < 2$. The surface integral is

$$
\begin{aligned}
\int\!\!\int_S (\boldsymbol{\nabla} \times \boldsymbol{F}).d\boldsymbol{S} &= \int_{\phi=0}^{2\pi} \int_{\rho=0}^{2} \left(-3\rho^3\right) d\rho\, d\phi \\
&= \int_{\phi=0}^{2\pi} \left[-\frac{3}{4}\rho^4\right]_{\rho=0}^{2} d\phi \\
&= -24\pi.
\end{aligned}
$$

Now for the line integral. As we noted above the bounding curve is the circle of radius 2 in the $z = 4$ plane, with positive sense in the counter-clockwise direction about the z-axis. In fact, on the curve $\rho = 2$ and $z = 4$ so we have,

$$\begin{aligned}
\boldsymbol{r} &= 2(\cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}) + 4\boldsymbol{k} \\
d\boldsymbol{r} &= 2(-\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j})\,d\phi \\
\boldsymbol{F} &= 8\sin^3\phi\,\boldsymbol{i} - 8\cos^3\phi\,\boldsymbol{j},
\end{aligned}$$

on $\partial\mathcal{S}$. Our integrand is

$$\boldsymbol{F}.d\boldsymbol{r} = -16(\sin^4\phi + \cos^4\phi)\,d\phi.$$

We need the following trig identities:

$$\begin{aligned}
\sin^2\phi &= \frac{1}{2}\left[1 - \cos(2\phi)\right] \\
\cos^2\phi &= \frac{1}{2}\left[1 + \cos(2\phi)\right].
\end{aligned}$$

So that,

$$\begin{aligned}
\sin^4\phi + \cos^4\phi &= \frac{1}{4}\left[2 + 2\cos^2(2\phi)\right] \\
&= \frac{1}{4}\left[3 + \cos(4\phi)\right],
\end{aligned}$$

where the double angle formula has been used for a second time. Finally, we have

$$\begin{aligned}
\int_{\partial\mathcal{S}} \boldsymbol{F}.d\boldsymbol{r} &= -4\int_{\phi=0}^{2\pi}\left[3 + \cos(4\phi)\right]\,d\phi \\
&= -24\pi.
\end{aligned}$$

We have the left and right sides equal and so we have verified Stokes' theorem in this case.

III. VARIATION IN TIME

16 Lecture 13

In any realistic mathematical applications we have to come to terms with the dynamic nature of our world. Most measurable quantities appear to vary in time. Our mathematical armoury has to reflect this.

Time Rates of Change of Field Quantities

Let P be a point in $\mathbb{R}^3$ moving along a differentiable curve $\mathcal{C}$,

$$\begin{aligned} x &= x(t) \\ y &= y(t) \\ z &= z(t), \end{aligned}$$

parametrised by the time, t. Suppose f is a quantity of physical interest, then in general f will vary in time in two ways. Firstly, there is an implicit dependence on time through the coordinates as P moves along its path. As P moves the coordinates change and so induce change in f. Secondly, there is an explicit time dependence, f may depend directly on time. These two forms of time dependence are evident, for example, in the temperature measured in a room over a period of time: If we follow a particular "air molecule" around the room the temperature changes as the molecule moves convectively about the room, but there is also the explicit time dependence of the room temperature – heating as the Sun rises, cooling as the Sun sets. We have $f = f(x, y, z, t)$ with x, y and z depending on t. To find the total time rate of change of f we simply use the chain rule of differentiation:

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= \frac{\partial f}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla} f \\ \text{where} \quad \boldsymbol{v} &= \frac{d\boldsymbol{r}}{dt}, \quad \text{the velocity of} \quad P. \end{aligned}$$

The same arguments apply to vector functions $\boldsymbol{F}$,

$$\frac{d\boldsymbol{F}}{dt} = \frac{\partial \boldsymbol{F}}{\partial t} + (\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{F}.$$

The operator,

$$\diamond \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v}.\boldsymbol{\nabla},$$

is said to give the *total rate of change in time.* This operator is also sometimes said to give the particle rate of change.

$$\frac{\partial}{\partial t} \quad \text{gives the } \textit{explicit} \text{ rate of change.}$$

$$\boldsymbol{v}.\boldsymbol{\nabla} \quad \text{gives the } \textit{convective} \text{ rate of change.}$$

Time Rate of Change of a Volume Integral

In our applications to fluid mechanics rates of change of bulk properties will be important – the time rate of change of the mass of fluid in a given region for example. In this section we develop the necessary formula to cover such situations.

Consider a closed surface $\mathcal{S}$, with no handles or self-intersections. We are going to allow this surface to distort in time, so its volume $V = V(t)$ will be a function of time. The picture we have in mind here is of a fluid with the individual "molecules" in $\mathcal{S}$ tagged, these marked molecules define the surface $\mathcal{S}$ and the volume it contains. We follow these molecules as they move in the body of the fluid. They move at different velocities so the surface begins to distort. Let each point $P = (x, y, z)$ move with velocity $\boldsymbol{v} = \boldsymbol{v}(x, y, z, t)$ – different points will move with different velocities, in general, as $\boldsymbol{v}$ is a vector function of position and time. At time $t + \delta t$, we assume δt small, P has moved to a new position. Call it P' with displacement vector $\overrightarrow{PP'} = \boldsymbol{v}\delta t$. By time $t + \delta t$ the points of $\mathcal{S}$ have moved to a new closed surface $\mathcal{S}'$, containing a volume $V' = V(t + \delta t)$. Let $f = f(x, y, z, t)$ be a differentiable function in a region containing $\mathcal{S}$ and $\mathcal{S}'$. Consider

$$I(t) = \iiint_{V(t)} f(x, y, z, t)\, dV.$$

We want to see how this quantity varies in time. We want to find dI/dt. Now

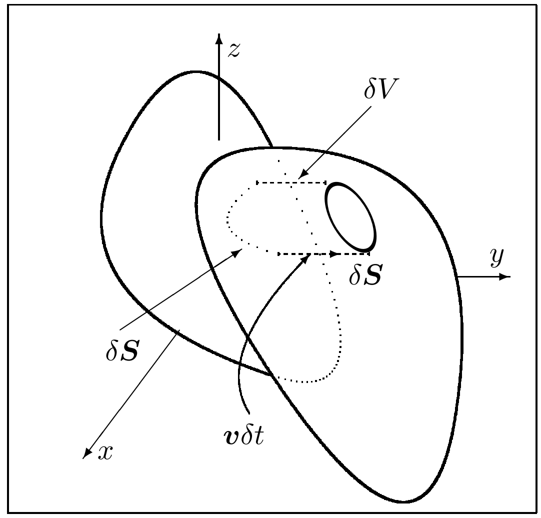
$$\begin{aligned}
I(t + \delta t) &= \iiint_{V(t+\delta t)} f(x, y, z, t + \delta t)\, dV \\
&= \iiint_{V(t)} f(x, y, z, t + \delta t)\, dV + \iiint_{\Delta V} f(x, y, z, t + \delta t)\, dV \\
\text{where} \quad \Delta V &= V(t + \delta t) - V(t).
\end{aligned}$$

Consider

$$\iiint_{\Delta V} f(x, y, z, t + \delta t)\, dV.$$

It will be a sum of terms of the form $f(x, y, z, t + \delta t)\, \delta V$ where δV is an element of the volume ΔV.

The volume δV is generated when the oriented surface element $\delta \boldsymbol{S}(t)$ moves from $\mathcal{S}$ to $\mathcal{S}'$ to give $\delta \boldsymbol{S}(t+\delta t)$. In fact, $\delta V = (\boldsymbol{v}\delta t).\delta \boldsymbol{S}(t)$ to the first order in δt. So, $f(x, y, z, t+\delta t)\, \delta V = f(x, y, z, t + \delta t)(\boldsymbol{v}.\delta \boldsymbol{S})\, \delta t$. Summing over all elements δV in ΔV and passing to the limit, we get



$$\int\int\int_{\Delta V} f(x, y, z, t + \delta t)\, dV = \int\int_{\mathcal{S}} f(x, y, z, t + \delta t)\boldsymbol{v}.d\boldsymbol{S}\, \delta t.$$

Returning to our expression for $I(t + \delta t)$ we have

$$I(t + \delta t) = \int\int\int_{V(t)} f(x, y, z, t + \delta t)\, dV + \delta t \int\int_{S(t)} f(x, y, z, t + \delta t)\boldsymbol{v}.d\boldsymbol{S}.$$

Hence,

$$\frac{I(t + \delta t) - I(t)}{\delta t} = \int\int\int_{V(t)} \left[\frac{f(x, y, z, t + \delta t) - f(x, y, z, t)}{\delta t}\right] dV + \int\int_{S} f(x, y, z, t+\delta t)\boldsymbol{v}.d\boldsymbol{S}.$$

Letting $\delta t \longrightarrow 0$ we get

$$\frac{dI}{dt} = \int\int\int_{V} \frac{\partial f}{\partial t}\, dV + \int\int_{S} f\, \boldsymbol{v}.d\boldsymbol{S}.$$

Using Gauss' theorem we have that

$$\int\int_{S} f\, \boldsymbol{v}.d\boldsymbol{S} = \int\int\int_{V} \boldsymbol{\nabla}.(f\boldsymbol{v})\, dV.$$

And we arrive at our result,

$$\begin{aligned} \diamond \quad \text{For} \quad I &= \int\int\int_{V(t)} f(x, y, z, t)\, dV \\ \frac{dI}{dt} &= \int\int\int_{V(t)} \left[\frac{\partial f}{\partial t} + \boldsymbol{\nabla}.(f\boldsymbol{v})\right] dV \\ &= \int\int\int_{V(t)} \left[\frac{df}{dt} + f\boldsymbol{\nabla}.\boldsymbol{v}\right] dV. \end{aligned}$$

Notice in the last line we have used the definition of the total derivative.

IV. APPLICATIONS TO FLUID MECHANICS

17 Lecture 14

Basic Ideas

A real fluid is composed of molecules interacting with each other via collisions and complex forces, e.g. Van der Waal's forces. To make progress in the description of fluids "in the large", i.e. at scales much larger than the molecular, we will need to introduce some simplifying assumptions. We would experience more than a little difficulty in trying to derive this bulk behaviour from molecular properties!

The basic idealisation we shall make is the *continuum hypothesis*. We will assume that the fluid is composed of indivisible matter. This assumption enables us to use that most powerful mathematical tool, the calculus. It allows us to ascribe to each point in space functions which describe the fluid. We can take limits and derivatives of these quantities without worrying about the underlying discrete molecular structure. Of course this is only an approximation, after all the fluid *is* composed of molecules. But it is a very good idealisation applying on scales from the very large to the very small.

We attribute to each point in the fluid, at each instant in time, the following properties:

- A *velocity* vector $\boldsymbol{v} = \boldsymbol{v}(x, y, z, t)$.
- A *density* $\mu = \mu(x, y, z, t)$.
- A *pressure* $p = p(x, y, z, t)$.

In general a fluid also possesses *viscosity* or (internal) friction. In this course, however we shall concern ourselves only with *inviscid* or frictionless fluids.

The equations which connect our three descriptive quantities are non-linear partial differential equations. These equations have proved extremely hard to solve in all but the simplest cases. Indeed, much recent work in fluid mechanics has been in the area of numerical integration of the equations. The equations and their generalisations which incorporate thermodynamic variables, such as temperature, are very important to many aspects of modern society: Aerodynamics of flight, the flow of water, weather forecasting, the behaviour of oils and lubricants and so on. Weather forecasting, usually done on a Cray supercomputer, uses a fluid model of the atmosphere.

The difficulty of the equations forces us to look at idealised situations - problems distinguished by their symmetry and (or) special conditions on the variables $\boldsymbol{v}$, μ and p. The four most common assumptions made are the following:

◇ *Inviscid* flow. We assume this throughout the course.

◇ *Incompressible* flow. All fluids are to some extent compressible, but under a large range of conditions many fluids can be approximated as fluids of constant density, i.e. as incompressible fluids. This approximation is particularly relevant for liquid, e.g. water, flow.

◇ *Steady* flow. The fluid quantities do not explicitly depend on time, i.e.

$$\frac{\partial \mu}{\partial t} = 0 = \frac{\partial p}{\partial t}, \quad \text{and} \quad \frac{\partial \boldsymbol{v}}{\partial t} = \boldsymbol{0}.$$

The fluid may still be moving, but our fluid quantities can change in time only through convection.

◇ *Irrotational* (or potential) flow. The *vorticity vector* $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$ describes the local rotational motion of the fluid, if $\boldsymbol{\omega} = \boldsymbol{0}$ the motion is said to be irrotational. This means that the velocity vector is conservative; considerable simplifications in the equations result.

In experiments, the dynamics of fluid motion is often followed in two ways. First, the paths of individual fluid "particles" can be followed by radioactive labelling of molecules. These paths are known as *pathlines*. The second method is to take a snapshot of a dye release in the fluid. In this case the geometry of the dye in the fluid is determined by the velocity vector at a fixed time, the time of the snapshot. The curves which result are called *streamlines*. Both sets of curves are integral curves, related to the velocity vector. The pathlines will be curves parametrised by the time,

$$\begin{aligned} \boldsymbol{r} &= \boldsymbol{r}(t) \\ \text{Pathlines,} \quad \frac{d\boldsymbol{r}}{dt}(t) &= \boldsymbol{v}(\boldsymbol{r}(t), t). \end{aligned}$$

The streamlines are curves pictured at a particular instant in time, $t = t_0$ say. Suppose we parametrise these curves with a parameter s. Then

$$\boldsymbol{r} = \boldsymbol{r}(s)$$

$$\text{streamlines,} \quad \frac{d\boldsymbol{r}}{ds}(s) \quad = \quad \boldsymbol{v}(\boldsymbol{r}(s), t_0).$$

In general, these two types of curve will be quite different.

♦**Example** Let the velocity vector of a fluid be given by

$$\boldsymbol{v} = \cos t\, \boldsymbol{i} + \sin t\, \boldsymbol{j}$$

Find

(a) The pathlines.

(b) The streamlines.

Solution:
(a) The equations for the pathlines are

$$\frac{d\boldsymbol{r}}{dt}(t) = \cos t\, \boldsymbol{i} + \sin t\, \boldsymbol{j},$$

or, in component form

$$\begin{aligned} \frac{dx}{dt} &= \cos t \\ \frac{dy}{dt} &= \sin t \\ \frac{dz}{dt} &= 0. \end{aligned}$$

Integrate the x equation to get $x = x_0 + \sin t$. Integrate the y equation to get $y = y_0 - \cos t$. Finally, the z equation gives $z = z_0$. Here, the x_0, y_0 and z_0 are constants. The pathlines are circles, of radius 1, in planes parallel to the $x \sim y$ plane.

(b) The equations for the streamlines are, in component form,

$$\begin{aligned} \frac{dx}{ds} &= \cos t_0 \\ \frac{dy}{ds} &= \sin t_0 \\ \frac{dz}{ds} &= 0. \end{aligned}$$

These equations are easily integrated to give

$$\begin{aligned} x &= x_0 + s\, \cos t_0 \\ y &= y_0 + s\, \sin t_0 \\ z &= z_0. \end{aligned}$$

Note that
$$\frac{x - x_0}{\cos t_0} = \frac{y - y_0}{\sin t_0},$$
so the streamlines are straight lines in planes parallel to the $x \sim y$ plane.

The Equation of Continuity

Let Ω be a region, in $\mathbb{R}^3$, filled with fluid. We suppose that there are no sources or sinks in Ω, i.e. there are no places where the fluid is created or destroyed. Consider an arbitrary simply connected volume $V = V(t)$ in Ω (c.f. Lecture 13). The volume is to move with the fluid in such a way that all the fluid particles initially in V remain in V as we follow the motion. The total mass in V is
$$M = \iiint_{V(t)} \mu \, dV,$$
recall, density is "mass divided by volume" – here we are "summing" all the infinitesimal masses $\mu \delta V$. The mass M must remain constant in time as V always contains the same number of fluid particles. So we have, using our formula from Lecture 13,
$$\begin{aligned} \frac{dM}{dt} &= \frac{d}{dt} \iiint_{V(t)} \mu \, dV \\ &= \iiint_{V(t)} \left(\frac{d\mu}{dt} + \mu \boldsymbol{\nabla}\boldsymbol{.}\boldsymbol{v} \right) dV \\ &= 0. \end{aligned}$$

This equation must be true for every admissible volume V in Ω. This can only be so if the integrand itself vanishes, consequently
$$\diamond \quad \frac{d\mu}{dt} + \mu \boldsymbol{\nabla}\boldsymbol{.}\boldsymbol{v} = 0.$$
This is the *continuity equation.* Using our expression for the total derivative it can also be written as
$$\diamond \quad \frac{\partial \mu}{\partial t} + \boldsymbol{\nabla}\boldsymbol{.}(\mu \boldsymbol{v}) = 0.$$
This equation takes much simpler forms under two of our basic simplifying assumptions.

If the flow is incompressible we have
$$\text{incompressible flow} \quad \boldsymbol{\nabla}\boldsymbol{.}\boldsymbol{v} = 0.$$
If the flow is steady we get
$$\text{steady flow} \quad \boldsymbol{\nabla}\boldsymbol{.}(\mu \boldsymbol{v}) = 0.$$

18 Lecture 15

Euler's Equation of Motion

To obtain a full description of a fluid we need to know the functional form of five scalar quantities, the three components of $\boldsymbol{v}$, μ and p. So far we have one scalar equation, the continuity equation. We need four more equations. Three of these equations will come in the form of one vector equation, the fluid analogue of "$\boldsymbol{F} = m\boldsymbol{a}$ ". That is, Newton's second law applied to fluids, in fact the goal of this section is to find this equation. We do not have to worry about dissipative frictional forces because we are assuming the fluid to be frictionless, i.e. inviscid. The remaining equation is known as an equation of state, an equation connecting the pressure p and the density μ. In fact, we have already met the simplest possible equation of state, μ =constant (incompressible flow). Generally, an equation of state is of the form $\mu = \mu(p)$. For an incompressible fluid μ is simply the constant function.

In applying the second law, "force equals mass times acceleration", we will need to resort again to infinitesimals and "sums" using integrals. The "force" part of the equation will be made up of two parts. External applied forces, such as gravitation, are known as *body forces*. Internal forces, exerted by the fluid on the fluid itself (forces due to pressure). As in our previous sections we will consider a simply connected volume $V = V(t)$ in a fluid region $\Omega \subseteq \mathbb{R}^3$, we will assume V to be bounded by the closed surface $\mathcal{S} = \mathcal{S}(t)$. We first want to find the total force acting on V, we will then equate this to the total "mass times acceleration" of V.

The body forces are easily dealt with. Let $\boldsymbol{F}$ be the *body force per unit mass* at each point in Ω. Then the total external force on V is,

$$\text{total body force on } V \; = \iiint_{V(t)} \boldsymbol{F}\mu \, dV.$$

We now examine the effect of the pressure. The naive definition of pressure is "force per unit area". We will have to sum the pressure over the entire surface $\mathcal{S}$ to get the total effect on V. Let δS be a small element of $\mathcal{S}$ with its outward unit normal being $\hat{\boldsymbol{n}}$. The force on δS due to the surrounding fluid is entirely due to pressure (remember δS moves with the fluid). The force is normal to the surface and acts inward – the crushing force you feel if you dive into deep enough water. So the force on δS is $-p\hat{\boldsymbol{n}}\,\delta S$. The total force on $\mathcal{S}$ due to the surrounding fluid will be,

$$\text{total pressure force on } V \; = -\iint_{\mathcal{S}} p \, d\boldsymbol{S} = -\iiint_{V(t)} (\boldsymbol{\nabla} p) \, dV,$$

where we have used a result derived from Gauss' theorem – see Assignment 3, Question 2.

The total force on V is the sum of the above two volume integrals,

$$\text{total force on } V \; = \iiint_V (\boldsymbol{F}\mu - \boldsymbol{\nabla} p)\; dV$$

Consider a small element δV of V. The mass of the element is $\mu\,\delta V$, so its "mass times acceleration" is

$$(\mu\,\delta V)\frac{d\boldsymbol{v}}{dt}.$$

So the total rate of change of linear momentum for V, its total "mass times acceleration", is

$$\iiint_V \frac{d\boldsymbol{v}}{dt}\mu\, dV.$$

Applying Newton's second law we have

$$\iiint_V \mu\frac{d\boldsymbol{v}}{dt}\, dV = \iiint_V (\boldsymbol{F}\mu - \boldsymbol{\nabla} p)\, dV.$$

This is

$$\iiint_V \mu\left[\frac{d\boldsymbol{v}}{dt} - \left(\boldsymbol{F} - \frac{\boldsymbol{\nabla} p}{\mu}\right)\right]\, dV = \boldsymbol{0}.$$

This equation must be true for every admissible volume V in Ω. Consequently we have

$$\diamond \quad \frac{d\boldsymbol{v}}{dt} = \boldsymbol{F} - \frac{1}{\mu}\boldsymbol{\nabla} p,$$

This is *Euler's equation* of motion for the fluid. Another very useful form of the equation is found by using our expression for the total derivative

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{v}.$$

The last term on the right hand side of the equation can be re-written using identity 6 of Lecture 3, with $\boldsymbol{F} = \boldsymbol{G} = \boldsymbol{v}$ and $\boldsymbol{v}.\boldsymbol{v} = v^2$,

$$\boldsymbol{\nabla} v^2 = 2(\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{v} + 2\boldsymbol{v}\times(\boldsymbol{\nabla} \times \boldsymbol{v}).$$

Introducing the *vorticity vector*

$$\diamond \quad \boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}.$$

We have,

$$(\boldsymbol{v}.\boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{v} + \boldsymbol{\nabla}(\frac{1}{2}v^2).$$

So that

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}(\frac{1}{2}v^2) + \boldsymbol{\omega} \times \boldsymbol{v}.$$

Put this into Euler's equation to get

$$\diamond \quad \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} = \boldsymbol{F} - \frac{1}{\mu}\boldsymbol{\nabla} p - \boldsymbol{\nabla}(\frac{1}{2}v^2).$$

This entirely equivalent to the original form of Euler's equation. It contains no new information, but it does have a very useful form.

In many problems of physical interest the body force present is a conservative force (e.g. gravitation); $\boldsymbol{F} = -\boldsymbol{\nabla} K$, where K is the potential energy for the force (the minus sign is traditional, it gives a positive potential energy when the zero of the potential energy is taken to be "infinity"). If an equation of state is specified, $\mu = \mu(p)$, then we have

$$\frac{1}{\mu}\boldsymbol{\nabla} p = \boldsymbol{\nabla}\left(\int \frac{dp}{\mu}\right),$$

where we have used a form of the chain rule, $\boldsymbol{\nabla} f(u) = f'(u)\boldsymbol{\nabla} u$. The second form of Euler's equation can now be written as,

$$\diamond \quad \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} = -\boldsymbol{\nabla}\left(K + \int \frac{dp}{\mu} + \frac{1}{2}v^2\right)$$

This is a very useful form of Euler's equation, particularly when one of our basic simplifying assumptions is applied.

$\diamond$ If the flow is incompressible, $\int \frac{dp}{\mu} = p/\mu$, Euler's equation gives

$$\text{incompressible flow,} \quad \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} = -\boldsymbol{\nabla}\left(K + \frac{p}{\mu} + \frac{1}{2}v^2\right).$$

$\diamond$ If the flow is steady we have

$$\text{steady flow,} \quad \boldsymbol{\omega} \times \boldsymbol{v} = -\boldsymbol{\nabla}\left(K + \int \frac{dp}{\mu} + \frac{1}{2}v^2\right).$$

$\diamond$ If the flow is irrotational then $\boldsymbol{\omega} = \boldsymbol{0}$ and there exists a *velocity potential*, ψ , such that $\boldsymbol{v} = \boldsymbol{\nabla}\psi$. Euler's equation becomes

$$\frac{\partial}{\partial t}(\boldsymbol{\nabla}\psi) = -\boldsymbol{\nabla}\left(K + \int \frac{dp}{\mu} + \frac{1}{2}v^2\right).$$

If we assume that $\psi = \psi(x, y, z, t)$ is a C^2 function then we can interchange the order of derivatives to get, after re-arranging the equation,

$$\boldsymbol{\nabla}\left(\frac{\partial \psi}{\partial t} + K + \int \frac{dp}{\mu} + \frac{1}{2}v^2\right) = 0.$$

The function in brackets cannot be a function of x, y or z as all these derivatives vanish (it has vanishing gradient). This expression must be a function of t alone, i.e.

$$\diamond \quad \frac{\partial\psi}{\partial t} + K + \int \frac{dp}{\mu} + \frac{1}{2}v^2 = f(t),$$

for some function of a single variable, f. This is *Bernoulli's equation* for irrotational flow with conservative body forces. If the flow is also incompressible, then

$$\diamond \quad \frac{\partial\psi}{\partial t} + K + \frac{p}{\mu} + \frac{1}{2}v^2 = f(t),$$

Bernoulli's equation for irrotational, incompressible flow with conservative body forces.

If, on the other hand, the flow is steady (and irrotational) we get

$$\diamond \quad K + \int \frac{dp}{\mu} + \frac{1}{2}v^2 = C,$$

where C is a constant.

A case which we will frequently encounter is steady, incompressible, irrotational flow with conservative body forces. In this case the continuity equation implies $\boldsymbol{\nabla}.\boldsymbol{v} = 0$, but we know $\boldsymbol{v} = \boldsymbol{\nabla}\psi$ so we see that ψ must be harmonic. In this case then we must first find the appropriate solution to Laplace's equation and then use Bernoulli's equation to determine the pressure. The problem would then be solved.

Incompressible, Irrotational, Steady Flow

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{\nabla}\psi \\ \nabla^2\psi &= 0 \\ K + \frac{1}{2}v^2 + \frac{p}{\mu} &= C. \end{aligned}$$

19 Lecture 16

Boundary Conditions

In most applications we are concerned with a fluid (gas or liquid) which is constrained in some way. The boundary could be an impermeable solid (e.g. a pipe or container) or another fluid, sometimes the boundary may be *free* so that the fluid forms a surface. In some mathematical idealisations we may wish to look at fluids of infinite extent. In this case we may want to impose conditions on the fluid at "infinity". Such conditions imposed at an interface are known as boundary conditions for the partial differential equations, or simply as boundary conditions. Two of these boundary conditions can be given a general mathematical formulation, they are the impermeable boundary condition and the free boundary condition.

Impermeable Boundaries

If a fluid is to move from one side of a surface to another it must have a non-zero component of velocity in the direction normal to that surface – tangential components of velocity will merely move the fluid along the surface. If the surface is impermeable to the fluid, then the fluid may not cross the surface. This means that the normal component of the fluid's velocity must be zero, relative to the surface. If the surface is moving with velocity $\boldsymbol{u}$ then we require that the normal component of $\boldsymbol{v} - \boldsymbol{u}$ must be zero on the surface. There will be no other boundary conditions on the fluid at this surface – we have assumed the fluid inviscid so there can be no tangential drag. Let the normal to the impermeable surface $\mathcal{S}$ be $\hat{\boldsymbol{n}}$, then we have

$$\text{impermeable boundary condition,} \quad [(\boldsymbol{v} - \boldsymbol{u}).\hat{\boldsymbol{n}}]_{\mathcal{S}} = 0.$$

Of course, if the boundary is stationary we simply have $[\boldsymbol{v}.\hat{\boldsymbol{n}}]_{\mathcal{S}} = 0$.

Free Surfaces

A free surface is the interface between a fluid and a vacuum, although sometimes we may use the condition as a good approximation to a situation where there is an interface between a liquid and a gas. We saw earlier that a surface in a fluid is, generally, subject to the internal force of pressure. If the free surface $\mathcal{S}$ were subject to this force, however, there would be a resultant acceleration and consequent distortion of the surface, i.e. $\mathcal{S}$ would not be the free surface. So the free surface condition is simply the vanishing of the pressure at the surface,

$$\text{free surface condition,} \quad p|_{\mathcal{S}} = 0.$$

This simply means that the vanishing of the pressure defines the free surface.

♦**Example** Consider an infinite, homogeneous atmosphere in static equilibrium above a plane surface. The equation of state is taken to be $p = C\mu^2$, where C is a constant. The only external force acting is the constant force of gravitation, with acceleration due to gravity g. Suppose the pressure at the surface is p_0, find the expression for the pressure in terms of the height above the plane and hence the depth of the atmosphere.

Solution: We choose Cartesian coordinates so that the plane surface is the $x \sim y$ plane, with the positive z axis measuring the height above the plane. Static equilibrium means that the acceleration vector of the fluid vanishes, from the original form of Euler's equation we get

$$\boldsymbol{F} - \frac{1}{\mu}\boldsymbol{\nabla} p = \boldsymbol{0}.$$

The body force per unit mass is, in this case, gravitation with $\boldsymbol{F} = -g\boldsymbol{k}$. The component form of our equation is

$$\begin{aligned} \frac{\partial p}{\partial x} &= 0 \\ \frac{\partial p}{\partial y} &= 0 \\ \frac{1}{\mu}\frac{\partial p}{\partial z} &= -g. \end{aligned}$$

So p (and hence μ) is a function of z only. Now, from the equation of state, we have

$$\frac{\partial p}{\partial z} = 2C\mu\frac{\partial \mu}{\partial z}.$$

Hence,

$$\frac{\partial \mu}{\partial z} = -\frac{g}{2C}.$$

Integrating, we get

$$\mu = C_0 - \frac{g}{2C}z,$$

where C_0 is a constant of integration. From this we get the following expression for p (using the equation of state),

$$p = C(C_0 - \frac{g}{2C}z)^2.$$

Next we use the condition $p(z = 0) = p_0$. Substituting $z = 0$ into our expression for p we have $p_0 = CC_0^2$ or $C_0 = \sqrt{\frac{p_0}{C}}$. So our expression for p is

$$p = C\left(\sqrt{\frac{p_0}{C}} - \frac{g}{2C}z\right)^2.$$

The upper boundary of the atmosphere will be a free surface, so $p(z = h) = 0$, where h is the depth of the atmosphere. This implies,

$$\sqrt{\frac{p_0}{C}} - \frac{g}{2C} h = 0,$$

from which we obtain a depth of $h = \frac{2\sqrt{Cp_0}}{g}$.

♦**Example** Consider the steady, irrotational, incompressible motion of a fluid in a vat with square base, with the base sides having length 1. Suppose that the velocity potential is given by

$$\psi = \frac{1}{2} V \cos(\pi x)\, \cos(\pi y) \left(e^{\sqrt{2}\pi z} + e^{-\sqrt{2}\pi z} \right).$$

We assume there are no external body forces. The sides of the vat are at $x = 0$, $x = 1$, $y = 0$ and $y = 1$; while the bottom is $z = 0$.

(a) Show that ψ does indeed satisfy Laplace's equation.

(b) Find the velocity of the flow.

(c) Show that the boundary conditions on the sides and bottom of the vat are satisfied.

(d) Find the equation of the surface of the fluid.

Solution: (a) Exercise! Just substitute ψ into Laplace's equation.
(b) The velocity is given by $\boldsymbol{v} = \boldsymbol{\nabla}\psi$, explicitly we have

$$\begin{aligned} \boldsymbol{v} &= \frac{1}{2} V \pi \Big[-\sin(\pi x)\, \cos(\pi y) \left(e^{\sqrt{2}\pi z} + e^{-\sqrt{2}\pi z} \right) \boldsymbol{i} - \cos(\pi x)\, \sin(\pi y) \left(e^{\sqrt{2}\pi z} + e^{-\sqrt{2}\pi z} \right) \boldsymbol{j} \\ &\quad + \sqrt{2} \cos(\pi x)\, \cos(\pi y) \left(e^{\sqrt{2}\pi z} - e^{-\sqrt{2}\pi z} \right) \boldsymbol{k} \Big]. \end{aligned}$$

(c) The boundary condition at $x = 0$ is $\boldsymbol{i}.\boldsymbol{v}|_{x=0} = 0$, we have

$$\boldsymbol{i}.\boldsymbol{v} = -\frac{1}{2} V \pi \sin(\pi x)\, \cos(\pi y) \left(e^{\sqrt{2}\pi z} + e^{-\sqrt{2}\pi z} \right),$$

which clearly vanishes at $x = 0$. This expression will also vanish at $x = 1$, so the boundary condition at $x = 1$ is satisfied as well. The other three conditions are similarly satisfied. You should check this.
(d) Bernoulli's equation in this case is $\frac{1}{2}v^2 + \frac{p}{\mu} = C$. From which we get the pressure

$$\begin{aligned} p &= \mu C - \mu \pi^2 \frac{1}{4} V^2 \Big[\sin^2(\pi x)\, \cos^2(\pi y) \left(e^{\sqrt{2}\pi z} + e^{-\sqrt{2}\pi z} \right)^2 \\ &\quad + \cos^2(\pi x)\, \sin^2(\pi y) \left(e^{\sqrt{2}\pi z} + e^{-\sqrt{2}\pi z} \right)^2 + \cos^2(\pi x)\, \cos^2(\pi y)\, 2 \left(e^{\sqrt{2}\pi z} - e^{-\sqrt{2}\pi z} \right)^2 \Big]. \end{aligned}$$

The "shape" of the surface of the fluid is determined by the free surface boundary condition, $p = 0$ at the free surface. Which gives

$$\pi^2 \frac{1}{4} V^2 \left[\sin^2(\pi x)\, \cos^2(\pi y)\, \left(e^{\sqrt{2}\pi z} + e^{-\sqrt{2}\pi z} \right)^2 + \cos^2(\pi x)\, \sin^2(\pi y)\, \left(e^{\sqrt{2}\pi z} + e^{-\sqrt{2}\pi z} \right)^2 \right.$$
$$\left. + \cos^2(\pi x)\, \cos^2(\pi y)\, 2 \left(e^{\sqrt{2}\pi z} - e^{-\sqrt{2}\pi z} \right)^2 \right] = C.$$

A very complex surface indeed! In fact, you may wonder how a vat of liquid could maintain such an unusual shape. The answer is found by looking at the pressure at the base of the vat, $z = 0$,

$$p = \mu C - \mu \pi^2 \frac{1}{4} V^2 \left[\sin^2(\pi x)\, \cos^2(\pi y) + \cos^2(\pi x)\, \sin^2(\pi y) \right].$$

So there is a rather unrealistic pressure distribution, on the liquid, at the base of the tank. This pressure would have to be maintained by some external agent. It is this which distorts the surface.

20 Lecture 17

Solving Problems In Fluid Dynamics

The following represents a list of key things which you must consider when you approach a problem in fluid dynamics. It is not meant to an exhaustive list of things you must do, nor a prescription for getting the right answer. Individual problems require individual solutions. The order in which you attack things has to be tailored to the problem at hand.

Read the problem carefully, listing any key words (e.g. steady, irrotational or incompressible).

Write down the continuity equation (if $\boldsymbol{\omega} \neq \mathbf{0}$) or Laplace's equation for the potential (if $\boldsymbol{\omega} = \mathbf{0}$). Write down the appropriate form of the Euler equation (if $\boldsymbol{\omega} \neq \mathbf{0}$) or Bernoulli's equation (if $\boldsymbol{\omega} = \mathbf{0}$).

Draw a sketch emphasising the symmetries of the problem.

Look for any simplifying assumptions which can be made due to particular symmetries of the problem. Choose your coordinates and axes to exploit these symmetries. Write down your reduced equations, i.e. the equations under these simplifying assumptions.

List the boundary conditions and write them in a form consistent with your symmetries and simplifying assumptions.

Solve the equations subject to the boundary conditions.

Answer the questions posed in the problem!

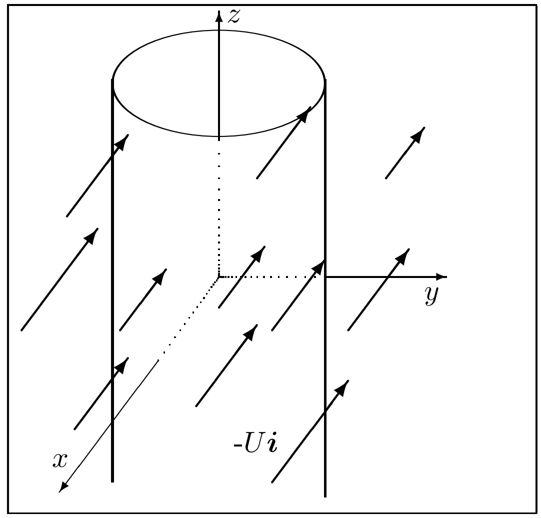
♦Example

Uniform Motion Past an Infinite Cylinder

Consider the steady, irrotational, incompressible flow of a fluid past an infinite, right circular cylinder of radius a. At large distances from the cylinder we assume the motion to be uniform (i.e. have constant velocity) in a direction perpendicular to the cylinder axis. The pressure at large distances is assumed constant. Neglecting body forces find the pressure p and the velocity $\boldsymbol{v}$ for the flow.

Solution: The key words tell us that the appropriate set of equations is just that listed at the end of Lecture 15. We choose the Cartesian axes so that the cylinder axis coincides with the z axis and so that the uniform flow at large distances is in the negative x direction.

As the cylinder is of infinite extent in the z direction the motion will look exactly the same in any plane parallel to the $x \sim y$ plane. This means that the fluid variables must be independent of z, this is a symmetry. Clearly cylindrical polar coordinates best exploit the symmetry of this problem.



At large distances from the cylinder, i.e. as $\rho \longrightarrow \infty$, the flow is uniform in the negative x direction, i.e. $\boldsymbol{v} \longrightarrow -U\boldsymbol{i}$, where U is a constant. We also have $p \longrightarrow p_\infty$, p_∞ a constant, as $\rho \longrightarrow \infty$. The flow is steady and independent of z, so our fluid variables depend only on (ρ, ϕ). We have, as the flow is irrotational,

$$\boldsymbol{v} = \boldsymbol{\nabla}\psi \quad \text{and} \quad \nabla^2\psi = 0,$$

with $\psi = \psi(\rho, \phi)$. And, as the flow is incompressible, μ = constant. To take account of the fact that the flow is uniform at large distances we introduce a function $V = V(\rho, \phi)$ such that

$$\psi = -Ux + V = -U\rho\cos\phi + V.$$

Then, $\boldsymbol{v} = -U\boldsymbol{i} + \boldsymbol{\nabla}V$, which in our cylindrical coordinates gives

$$\boldsymbol{v} = -U\boldsymbol{i} + \hat{\boldsymbol{\rho}}\frac{\partial V}{\partial \rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial V}{\partial \phi}.$$

Using the fact that the normal to the cylinder surface is $\hat{\boldsymbol{\rho}}$ and that $\hat{\boldsymbol{\rho}}.\boldsymbol{i} = \cos\phi$, our boundary conditions are

$$\left.\hat{\boldsymbol{\rho}}.\boldsymbol{v}\right|_{\rho=a} = -U\cos\phi + \left.\frac{\partial V}{\partial \rho}\right|_{\rho=a} = 0$$

$$\frac{\partial V}{\partial \rho} \longrightarrow 0 \quad \text{and} \quad \frac{1}{\rho}\frac{\partial V}{\partial \phi} \longrightarrow 0 \text{ as } \rho \longrightarrow \infty.$$

This second pair of conditions just come from $\boldsymbol{v} \longrightarrow -U\boldsymbol{i}$ as $\rho \longrightarrow \infty$. We *assume* a separable solution for V, i.e. we take $V = R(\rho)T(\phi)$, and substitute into the first of the boundary conditions, to get

$$T(\phi)R'(a) = U\cos\phi \text{ or } T(\phi) = \frac{U}{R'(a)}\cos\phi.$$

We have found the ϕ dependence of the velocity potential. We now need to find the ρ dependence. We have thus far $\psi = -U\rho\cos\phi + \frac{UR(\rho)}{R'(a)}\cos\phi$, which we substitute into Laplace's equation (written in cylindrical coordinates, see Lecture 7), we have after cancelling the $U\cos\phi\,/R'(a)$

$$R'' + \frac{1}{\rho}R' - \frac{1}{\rho^2}R = 0.$$

Where the dashes indicate ρ differentiation. You should check this equation. This equation is solved via the substitution $R = u/\rho$, where $u = u(\rho)$. Making the substitution gives, after collecting together the terms of a product rule,

$$\left(\frac{u'}{\rho}\right)' = 0.$$

This is easily integrated to give

$$u = C_1\frac{1}{2}\rho^2 + C_2.$$

Whence,

$$V = \frac{U}{R'(a)}\cos\phi\left(\frac{1}{2}C_1\rho + \frac{C_2}{\rho}\right).$$

Applying the second pair of boundary conditions now gives $C_1 = 0$ and $C_2 = -a^2R'(a)$. We have now completely determined the velocity potential,

$$\psi = -U\rho\cos\phi - \frac{a^2U}{\rho}\cos\phi.$$

The velocity is

$$\boldsymbol{v} = \boldsymbol{\nabla}\psi = U\left(\frac{a^2}{\rho^2} - 1\right)\cos\phi\,\hat{\boldsymbol{\rho}} + U\left(\frac{a^2}{\rho^2} + 1\right)\sin\phi\,\hat{\boldsymbol{\phi}}.$$

Our final step is to find the pressure using the Bernoulli equation

$$\frac{1}{2}v^2 + \frac{p}{\mu} = C_0.$$

First we determine v^2,

$$v^2 = \boldsymbol{v.v} = U^2\left[\left(\frac{a^4}{\rho^4} + 1\right) - 2\frac{a^2}{\rho^2}\cos(2\phi)\right].$$

Applying the boundary conditions at infinity to Bernoulli's equation, $v^2 \longrightarrow U^2$ and $p \longrightarrow p_\infty$, gives

$$C_0 = \frac{1}{2}U^2 + \frac{p_\infty}{\mu}.$$

So we can now determine p from the Bernoulli equation,

$$p = p_\infty + \frac{1}{2}\mu U^2 \frac{a^2}{\rho^2}\left[2\cos(2\phi) - \frac{a^2}{\rho^2}\right].$$

The problem is solved! The details at some stages in the solution have been left out. You should try and fill in these gaps yourself.

This is quite a difficult and intricate problem to solve. But it illustrates all the steps necessary in solving a full boundary value problem. It is certainly a more difficult problem than you can expect in the examination! The assignments and tutorials are a better guide to the standard of examination problems.

21 Lecture 18

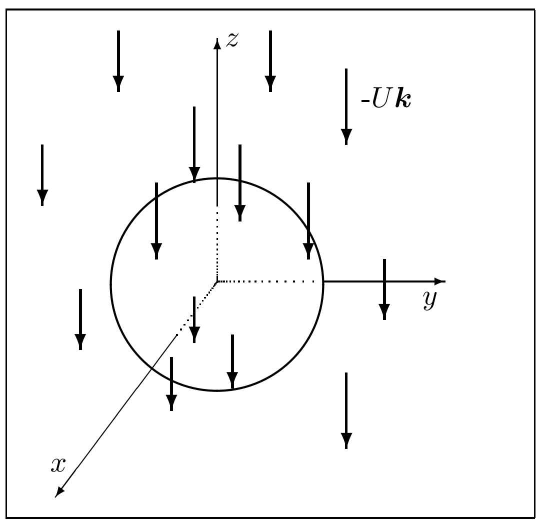
♦Example

Uniform Flow Past an Immersed Sphere

Consider the steady, irrotational, incompressible flow passed an immersed sphere of radius a. We assume the flow to be uniform at large distances from the sphere. The pressure at large distances is assumed constant and we neglect body forces. Determine the pressure and velocity of the flow. What is the thrust on the hemisphere facing the fluid flow? What is the thrust on the whole sphere?

Solution: Choose Cartesian axes with the sphere at the origin. Orient the z axis so that at large distances from the sphere the fluid velocity is $-U\boldsymbol{k}$, where U is a constant. We will use spherical polar coordinates, they are clearly well adapted to the symmetry of the problem.

We imitate our previous example and introduce a function V, with $\psi = -Uz + V$. The velocity vector is, $\boldsymbol{v} = -U\boldsymbol{k} + \boldsymbol{\nabla} V$. Then $\boldsymbol{\nabla} V \longrightarrow 0$, as $r \longrightarrow \infty$. The problem has axial symmetry about the $\boldsymbol{k}$ axis, i.e. the fluid motion (and hence the fluid variables) does not change as we rotate about the $\boldsymbol{k}$ axis.



This means the fluid variables are independent of the coordinate ϕ. The motion is also steady so that all fluid variables are functions of (r, θ) only. The boundary condition at the sphere's surface is (we assume the surface to be impermeable) $\hat{\boldsymbol{r}}.\boldsymbol{v}|_{r=a} = 0$. Which gives

$$\left.\frac{\partial V}{\partial r}\right|_{r=a} = U\cos\theta.$$

We look for a separable solution, $V = R(r)T(\theta)$, which gives after substitution into the above boundary condition

$$T(\theta) = \frac{U}{R'(a)}\cos\theta.$$

We have,

$$\psi = -Ur\cos\theta + \frac{U}{R'(a)}R(r)\cos\theta.$$

The motion is irrotational and incompressible so ψ must satisfy Laplace's equation, in spherical polars (see Lecture 8). After simplification, we get an equation for R,

$$r^2R'' + 2rR' - 2R = 0.$$

This equation is solved using the substitution $R = u(r)/r^2$, after substitution we get

$$u'' - \frac{2}{r}u' = 0.$$

Which can be written as

$$\left(\frac{u'}{r^2}\right)' = 0.$$

Integrating once we have $u' = C_1r^2$ and a final integration gives $u = \frac{1}{3}C_1r^3 + C_2$. So our function R is

$$R = \frac{1}{3}C_1r + \frac{C_2}{r^2}.$$

We now apply the boundary condition at infinity, $\boldsymbol{\nabla}V \longrightarrow 0$, as $r \longrightarrow \infty$. As $V = \frac{U}{R'(a)}(\frac{1}{3}C_1r + \frac{C_2}{r^2})\cos\theta$ we have

$$\left(\frac{1}{3}C_1 - \frac{2C_2}{r^3}\right)\cos\theta\,\hat{\boldsymbol{r}} - \frac{1}{r}\left(\frac{1}{3}C_1r + \frac{C_2}{r^2}\right)\sin\theta\,\hat{\boldsymbol{\theta}} \longrightarrow 0,$$

as $r \longrightarrow \infty$, where we have used the expression for the gradient in spherical polar coordinates (see Lecture 8). We must have $C_1 = 0$. Next apply our first boundary condition,

$$\left.\frac{\partial V}{\partial r}\right|_{r=a} = U\cos\theta,$$

to get

$$-\frac{2UC_2}{R'(a)a^3}\cos\theta = U\cos\theta.$$

This gives us C_2 as,

$$C_2 = -\frac{1}{2}a^3R'(a).$$

We have now fully determined the velocity potential as

$$\psi = -\frac{U}{2}r\left(2 + \frac{a^3}{r^3}\right)\cos\theta.$$

The velocity is now found as the gradient of this expression,

$$\boldsymbol{v} = -U\left(1 - \frac{a^3}{r^3}\right)\cos\theta\,\hat{\boldsymbol{r}} + U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta\,\hat{\boldsymbol{\theta}}.$$

From this we obtain

$$v^2 = U^2 \left[1 + \frac{a^3}{r^3}(1 - 3\cos^2\theta\,) + \frac{a^6}{4r^6}(1 + 3\cos^2\theta\,) \right].$$

Bernoulli's equation is,

$$\frac{p}{\mu} + \frac{1}{2}v^2 = C.$$

Applying the boundary condition at infinity to this equation gives

$$C = \frac{p_\infty}{\mu} + \frac{1}{2}U^2.$$

Putting in all the ingredients into the Bernoulli equation we can now determine the pressure,

$$p = p_\infty - \frac{1}{2}\mu U^2 \frac{a^3}{r^3} \left[(1 - 3\cos^2\theta\,) + \frac{a^3}{4r^3}(1 + 3\cos^2\theta\,) \right].$$

Finally, we are to determine the two thrusts. The thrust on an element $d\boldsymbol{S}$ of the surface of the sphere is $-pd\boldsymbol{S}$ at $r = a$. We need to integrate this expression, however we must take care with the coordinate dependent normal $\hat{\boldsymbol{r}} = \sin\theta\,\cos\phi\,\boldsymbol{i} + \sin\theta\,\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}$. The thrust on the half sphere is the vector force

$$-\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} p(a,\theta)(\sin\theta\,\cos\phi\,\boldsymbol{i} + \sin\theta\,\sin\phi\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k})a^2\sin\theta\,d\theta\,d\phi,$$

where we have used $dS = a^2\sin\theta\,d\theta\,d\phi$ for the sphere. We note that the coefficients of $\boldsymbol{i}$ and $\boldsymbol{j}$ will both vanish because the ϕ integrals of $\cos\phi$ and $\sin\phi$ both vanish between the limits $0 < \phi < 2\pi$. This leaves the integrals giving the coefficient of $\boldsymbol{k}$,

$$-a^2\boldsymbol{k}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} p(a,\theta)\cos\theta\,\sin\theta\,d\theta\,d\phi.$$

Doing the ϕ integration and substituting for p we have

$$-2\pi a^2\boldsymbol{k}\int_{\theta=0}^{\frac{\pi}{2}} \left[p_\infty - \frac{1}{8}\mu U^2\left(5 - 9\cos^2\theta\,\right) \right] \cos\theta\,\sin\theta\,d\theta\ .$$

Integrating,

$$-2\pi a^2\boldsymbol{k}\left\{ \left(p_\infty - \frac{5}{8}\mu U^2\right)\left[-\frac{1}{4}\cos(2\theta)\right]_{\theta=0}^{\frac{\pi}{2}} + \frac{9}{8}\mu U^2\left[-\frac{1}{4}\cos^4\theta\right]_{\theta=0}^{\frac{\pi}{2}} \right\}.$$

Evaluating, we find the thrust on the hemisphere as

$$-\pi a^2\left(p_\infty - \frac{1}{16}\mu U^2\right)\boldsymbol{k}.$$

The fact that the thrust is entirely in the $\boldsymbol{k}$ direction is what we would expect – thats the direction of the flow. Calculation of the thrust on the entire sphere follows the same pattern, except that the limits of the θ integration are now $0 < \theta < \pi$. The thrust is

$$-2\pi a^2 \boldsymbol{k} \left\{ \left(p_\infty - \frac{5}{8}\mu U^2 \right) \left[-\frac{1}{4}\cos(2\theta) \right]_{\theta=0}^{\pi} + \frac{9}{8}\mu U^2 \left[-\frac{1}{4}\cos^4\theta \right]_{\theta=0}^{\pi} \right\}.$$

Evaluating, we have the required thrust as zero! At first thought this does not seem correct, however the pressure is distributed symmetrically on either side of the $x \sim y$ plane. Notice that θ occurs in p only in the form $\cos^2\theta$ so that the pressure is reflection symmetric in the $x \sim y$ plane. As the pressure acts in a direction normal to the surface, along inward radial lines, the total pressure force on the sphere is thus zero.

22 Lecture 19

♦Example

<u>A Time Dependent Flow</u>

The radius R of an impermeable sphere immersed in an infinite ocean of incompressible fluid of density μ varies according to $R = A + a\cos nt$, where A, a and n are constants. The fluid moves irrotationally and radially with no body forces. The constant pressure at infinity is P.

(a) Show that the velocity potential may be taken as

$$\psi = \frac{F(t)}{r}, \quad F \text{ a function of } t \text{ only.}$$

(b) Obtain the pressure at any point on the surface of the sphere and show that the maximum pressure obtained is

$$p = P + \frac{n^2}{2}\mu\left(3a^2 + \frac{1}{5}A^2\right), \quad \text{provided } \ A \leq 5a.$$

<u>Solution</u>: The fluid is incompressible and moves irrotationally, so

$$\begin{aligned} \mu &= \text{constant} \\ \boldsymbol{v} &= \boldsymbol{\nabla}\psi \\ \nabla^2\psi &= 0 \end{aligned}$$

(a) The fluid moves radially so we have $\psi = \psi(r,t)$. Using spherical polar coordinates Laplace's equation for ψ is

$$\frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial r^2} = 0,$$

which can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = 0.$$

Integrating, and remembering that ψ is a function of r and t,

$$r^2\frac{\partial\psi}{\partial r} = -F(t).$$

Here F is an arbitrary function of t, it is the "integration constant". Dividing by r^2 and integrating again we have

$$\psi = \frac{F(t)}{r} + \text{arbitrary function of } t,$$

where we may drop the "integration constant" (arbitrary function of t) as ψ is only determined up to an additive arbitrary function of t. The potential is

$$\psi = \frac{F(t)}{r}.$$

(b) The velocity is

$$\boldsymbol{v} = \boldsymbol{\nabla}\psi = -F(t)\frac{\hat{\boldsymbol{r}}}{r^2}.$$

The boundary surface at $r = R(t)$ moves purely radially, with speed $R'(t) = -an\sin nt$. Applying the boundary condition at $r = R$ we have

$$[\boldsymbol{v}.\hat{\boldsymbol{r}} - (-an\sin nt)]_{r=R} = 0, \quad \text{where } \; R = A + a\cos nt.$$

From which we get

$$F(t) = an\sin nt\, R^2 = an\sin nt\,(A + a\cos nt\,)^2.$$

We can now apply Bernoulli's equation for incompressible, irrotational flow. We have

$$\frac{\partial\psi}{\partial t} + \frac{1}{2}v^2 + \frac{p}{\mu} = f(t).$$

Now

$$\begin{aligned} \boldsymbol{v} &= \boldsymbol{\nabla}\psi = -F(t)\frac{\hat{\boldsymbol{r}}}{r^2}, \\ \text{so } \; v^2 &= \frac{[F(t)]^2}{r^4}. \\ \text{And } \; \frac{\partial\psi}{\partial t} &= \frac{F'(t)}{r}, \end{aligned}$$

where F is given as above. Bernoulli's equation becomes

$$\frac{F'(t)}{r} + \frac{[F(t)]^2}{2r^4} + \frac{p}{\mu} = f(t).$$

Applying the boundary condition at infinity, $p \longrightarrow P$ as $r \longrightarrow \infty$, we have

$$f(t) = \frac{P}{\mu},$$

so f is a constant. Our Bernoulli equation now gives us p

$$p = P - \mu\left\{\frac{F'(t)}{r} + \frac{[F(t)]^2}{2r^4}\right\},$$

with F given above. On the surface of the sphere, $r = R$, so the pressure on the sphere's surface is

$$\begin{aligned} p &= P - \mu\left\{\frac{F'(t)}{R} + \frac{[F(t)]^2}{2R^4}\right\} \\ &= P - \mu\left\{an^2\cos nt\,(A + a\cos nt\,) - 2a^2n^2\sin^2 nt\, + \frac{1}{2}a^2n^2\sin^2 nt\right\} \\ \text{i.e. } \; p &= P + \mu\frac{an^2}{2}\left(3a - 2A\cos nt - 5a\cos^2 nt\right). \end{aligned}$$

To find the maximum of this expression we take its t derivative and equate to zero:

$$\text{for } n \neq 0, \;\; 2A \sin nt \; + 10a \cos nt \sin nt \; = 0.$$

There are two possibilities, $\sin nt \; = 0$ or $\cos nt \; = -\frac{A}{5a}$. The second of these possibilities corresponds to the maximum. Note that we must have $\cos nt \leq 1$, so $A \leq 5a$. Substituting $\cos nt \; = -\frac{A}{5a}$ into the expression for the pressure at the sphere's surface we get

$$\text{maximum } \; p = P + \frac{n^2}{2}\mu\left(3a^2 + \frac{1}{5}A^2\right),$$

as required.

Kelvin's Theorem

In most of our examples we have made the assumption that the fluid is irrotational. How realistic is such an assumption? For incompressible fluids we will see that under a large range of conditions it is a very good assumption. Kelvin's theorem shows that irrotational motion is the state of lowest kinetic energy. That is, the fluid can minimise its kinetic energy by moving irrotationally.

◇Theorem (Kelvin)

The irrotational motion of an incompressible fluid occupying a simply connected region $\Omega \subseteq \mathrm{I\!R}^3$ *has less kinetic energy than any other motion with the same boundary conditions.*

Proof: Suppose our fluid region is bounded by a closed surface $\partial\Omega$, with unit normal $\hat{\boldsymbol{n}}$. Let the normal component of the velocity at the boundary be given as $\boldsymbol{v}.\hat{\boldsymbol{n}} = V$. We will obtain an expression for the total kinetic energy of the fluid in the region Ω in both the irrotational and non-irrotational cases. First the irrotational case. We have $\boldsymbol{v} = \boldsymbol{\nabla}\psi$, so the total kinetic energy of the fluid in Ω is

$$\begin{aligned}
T_{irrot} &= \iiint_\Omega \frac{1}{2}\mu v^2 \, dV \\
&= \frac{1}{2}\mu \iiint_\Omega (\boldsymbol{\nabla}\psi)\boldsymbol{.}(\boldsymbol{\nabla}\psi)\, dV \\
&= \frac{1}{2}\mu \iiint_\Omega \left[\boldsymbol{\nabla .}(\psi\boldsymbol{\nabla}\psi) - \psi\nabla^2\psi\right] \, dV \\
&= \frac{1}{2}\mu \iiint_\Omega \boldsymbol{\nabla .}(\psi\boldsymbol{\nabla}\psi)\, dV, \;\; \text{as } \; \nabla^2\psi = 0. \\
&= \frac{1}{2}\mu \iint_{\partial\Omega} (\psi\boldsymbol{\nabla}\psi)\boldsymbol{.}\, d\boldsymbol{S}, \;\; \text{by Gauss' theorem.} \\
&= \frac{1}{2}\mu \iint_{\partial\Omega} \psi\boldsymbol{v}.\hat{\boldsymbol{n}}\, dS
\end{aligned}$$

$$\text{i.e. } T_{irrot} = \frac{1}{2}\mu \int\!\!\int_{\partial\Omega} \psi V \, dS.$$

We now treat the non-irrotational case. Let $\boldsymbol{u}$ be the velocity of this motion. Define $\boldsymbol{U}$ by

$$\boldsymbol{U} = \boldsymbol{u} - \boldsymbol{v},$$

where $\boldsymbol{v}$ is the irrotational fluid velocity, above. So we have $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{U} = \boldsymbol{\nabla}\psi + \boldsymbol{U}$. We assume this non-irrotational motion to have the same boundary values as the irrotational one. That is $\boldsymbol{u}.\hat{\boldsymbol{n}}|_{\partial\Omega} = \boldsymbol{v}.\hat{\boldsymbol{n}}|_{\partial\Omega}$. Consequently,

$$\boldsymbol{U}.\hat{\boldsymbol{n}}|_{\partial\Omega} = 0.$$

We also note that as both motions are incompressible we have $\boldsymbol{\nabla}.\boldsymbol{u} = 0$ and $\boldsymbol{\nabla}.\boldsymbol{v} = 0$ (from the continuity equation) , so that $\boldsymbol{\nabla}.\boldsymbol{U} = 0$. We now calculate the total kinetic energy of this motion.

$$\begin{aligned} T &= \frac{1}{2}\mu \int\!\!\int\!\!\int_{\Omega} \boldsymbol{u}.\boldsymbol{u} \, dV \\ &= \frac{1}{2}\mu \int\!\!\int\!\!\int_{\Omega} (\boldsymbol{v} + \boldsymbol{U}).(\boldsymbol{v} + \boldsymbol{U}) \, dV \\ &= \frac{1}{2}\mu \int\!\!\int\!\!\int_{\Omega} (v^2 + U^2 + 2\boldsymbol{v}.\boldsymbol{U}) \, dV \\ &= T_{irrot} + \frac{1}{2}\mu \int\!\!\int\!\!\int_{\Omega} (U^2 + 2\boldsymbol{v}.\boldsymbol{U}) \, dV, \end{aligned}$$

as the integral of U^2 is positive we have

$$T \geq T_{irrot} + \mu \int\!\!\int\!\!\int_{\Omega} \boldsymbol{v}.\boldsymbol{U} \, dV.$$

Now

$$\boldsymbol{v}.\boldsymbol{U} = (\boldsymbol{\nabla}\psi).\boldsymbol{U} = \boldsymbol{\nabla}.(\psi\boldsymbol{U}) - \psi\boldsymbol{\nabla}.\boldsymbol{U},$$

so we have

$$\begin{aligned} \mu \int\!\!\int\!\!\int_{\Omega} \boldsymbol{v}.\boldsymbol{U} \, dV &= \mu \int\!\!\int\!\!\int_{\Omega} [\boldsymbol{\nabla}.(\psi\boldsymbol{U}) - \psi\boldsymbol{\nabla}.\boldsymbol{U}] \, dV \\ &= \mu \int\!\!\int_{\partial\Omega} \psi\boldsymbol{U}.d\boldsymbol{S}, \text{ using Gauss' theorem and } \boldsymbol{\nabla}.\boldsymbol{U} = 0. \\ &= \mu \int\!\!\int_{\partial\Omega} \psi\boldsymbol{U}.\hat{\boldsymbol{n}} \, dS \\ &= 0, \text{ as } \boldsymbol{U}.\hat{\boldsymbol{n}}|_{\partial\Omega} = 0. \end{aligned}$$

We put this back into our inequality for T to get the desired result,

$$T \geq T_{irrot}.$$

Among all motions of an incompressible fluid the irrotational motion has the smallest kinetic energy.

V. FOURIER SERIES

23 Lecture 20

One of the most striking observations made about Nature is the plethora of periodic phenomena: the seasons, night and day, the tides, wave motion, vibrational modes in stringed instruments, vibrational modes of atoms and molecules, ... Our ability to qualitatively and quantitatively describe such behaviour is clearly of importance in both mathematics and physical theory. Our task in this section is introduce the necessary mathematical techniques for such a description. In fact, the theory turns out to have much wider applicability.

Periodic Functions and Fourier Series

◇ A function, f, on $\mathbb{R}$ is said to be *periodic* of period T if

$$f(x+T) = f(x), \quad \text{for all } \ x \in \mathbb{R}.$$

The functions $\sin x$, $\cos x$, $\sin nx$ and $\cos nx$ (n an integer) are all periodic of period 2π. The functions $\sin(2\pi x/T)$, $\cos(2\pi x/T)$, $\sin(2\pi nx/T)$ and $\cos(2\pi nx/T)$ (n an integer) are all periodic of period T, check the definition.

A natural question to ask is whether any periodic function can be written as a sum (possibly infinite) of sines and cosines. What this question is really asking is, do the sine and cosine functions form some sort of basis for the periodic functions? The answer turns out to be a qualified yes. How then is this series to be defined?

Let $f = f(x)$ be a periodic function of period 2π. We'll deal with more general periods later. There is nothing really new in taking period T – it's just a rescaling of the unit of length. Suppose f can be written as a series in sines and cosines, specifically

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \ + \sum_{n=1}^{\infty} b_n \sin nx \,, \end{aligned}$$

where the a_n and b_n are constants. Of course there are questions of validity and convergence to be looked at, but we will ignore these for the time being. What we want to do at the moment is find the series. This means finding the constants a_n and b_n. These constants are known as the *Fourier coefficients*. Note the factor of one-half multiplying the a_0, the reason for this will become apparent below.

To determine the coefficients we will need to make use of the following integrals

$$\begin{aligned}
\frac{1}{\pi}\int_{x=-\pi}^{\pi} \sin mx\, \sin nx\, dx &= \begin{cases} 0, & m=n=0, \\ 0, & m\neq n, \\ 1, & m=n\neq 0. \end{cases} \\
\frac{1}{\pi}\int_{x=-\pi}^{\pi} \cos mx\, \cos nx\, dx &= \begin{cases} 0, & m\neq n, \\ 2, & m=n=0, \\ 1, & m=n\neq 0. \end{cases} \\
\frac{1}{\pi}\int_{x=-\pi}^{\pi} \sin mx\, \cos nx\, dx &= 0, \ \ m \text{ and } n \text{ integers.}
\end{aligned}$$

Note the appearance of the 2 on the left hand side of the second equation, it is this factor which will explain the factor of one-half mentioned above.

The Fourier coefficients can now be determined. Firstly the a_n, using the series definition of f we have

$$\begin{aligned}
\frac{1}{\pi}\int_{x=-\pi}^{\pi} f(x)\cos mx\, dx &= \frac{a_0}{2\pi}\int_{-\pi}^{\pi}\cos mx\, dx + \sum_{n=1}^{\infty}\frac{a_n}{\pi}\int_{-\pi}^{\pi}\cos nx\, \cos mx\, dx \\
&\quad + \sum_{n=1}^{\infty}\frac{b_n}{\pi}\int_{-\pi}^{\pi}\sin nx\, \cos mx\, dx.
\end{aligned}$$

When $m=0$ we have , using our integrals,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = a_0.$$

For $m \geq 1$, we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\, dx = a_m.$$

In a similar vein,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\, dx = b_m.$$

We can now use our acquired hindsight to define the Fourier series of a function.

◇ *Fourier series.* Suppose $f=f(x)$ is a function given on $-\pi < x < \pi$ and defined to be periodic, of period 2π, on all of $\mathbb{R}$. Then we define the Fourier series of f as

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{with } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx\,, \quad n = 0,1,2,3,\ldots$$
$$\text{and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\,, \quad n = 1,2,3,\ldots.$$

Where $\sim$ is read as "has Fourier series". We don't use equality here as there may be points where the series does not converge or where it converges to a value different from $f(x)$.

♦**Example** Find the Fourier series of

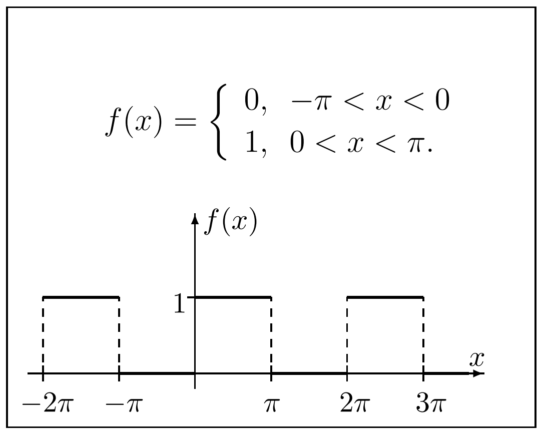
$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 < x < \pi, \end{cases}$$

where f is defined to be periodic of period 2π on $\mathbb{R}$.

Solution: We need to find the Fourier coefficients, a_n and b_n.

From the definition above we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0.\cos nx\,dx \right.\\ &\qquad \left. + \int_{0}^{\pi} 1.\cos nx\,dx\right) \\ &= \frac{1}{\pi}\left(\int_{0}^{\pi} \cos nx\,dx\right). \end{aligned}$$



There will be two cases to consider.

For $n = 0$ we have, $a_0 = 1$. For $n \geq 1$ we get,

$$a_n = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = 0.$$

For the coefficients b_n, remember $n \geq 1$, we have

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\ &= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0.\sin nx\,dx + \int_{0}^{\pi} 1.\sin nx\,dx\right) \\ &= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} \\ &= \frac{1}{n\pi}(1 - \cos n\pi\,) \\ &= \frac{1}{n\pi}[1 - (-1)^n], \end{aligned}$$

where we have used the identity $\cos n\pi = (-1)^n$. Our Fourier series is

$$f(x) \sim \frac{1}{2} + \frac{2}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots).$$

♦**Example** Find the Fourier series for $f(x) = x$.

Solution: To calculate the Fourier coefficients for this example we will need to use integration by parts. In fact, as you will see, most examples will require the use of integration by parts when calculating Fourier coefficients. For the a_n we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx.$$

As is usual there are two cases for the a_n, $n = 0$ and $n \geq 1$. For $n = 0$ we have,

$$\begin{aligned} a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx &= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} \\ &= 0. \end{aligned}$$

For $n \geq 1$,

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\left(\frac{\sin nx}{n}\right)' dx \\ &= \frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\sin nx}{n}\,dx\right\} \\ &= \frac{1}{n\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{\pi} \\ &= 0. \end{aligned}$$

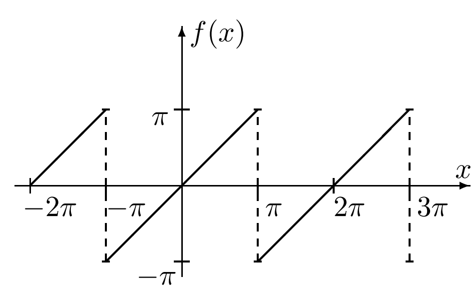
We have used the integration by parts formula

$$\int_{-\pi}^{\pi} uv'\,dx = [uv]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u'v\,dx$$

with $u = x$ and $v = (\sin nx)/n$. So all the a_n are zero. Now to the b_n.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\left(-\frac{\cos nx}{n}\right)' dx \\ &= \frac{1}{\pi}\left\{\left[-x\frac{\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}\,dx\right\} \\ &= -\frac{2}{n}\cos n\pi \\ &= \frac{2}{n}(-1)^{n+1}. \end{aligned}$$

$f(x) = x, \quad -\pi < x < \pi.$



We have our Fourier series,

$$x \sim 2\left\{\sin x \, - \frac{1}{2}\sin 2x \, + \frac{1}{3}\sin 3x \, - \frac{1}{4}\sin 4x \, + \cdots\right\}.$$

24 Lecture 21

The Dirichlet Conditions

The Dirichlet theorem (below) seeks to establish a class of functions for which the Fourier series is well defined and convergent, at least at points where the function is reasonably behaved. Before looking at this theorem it is perhaps worth while making contact with some of the theoretical background to your work in the practicals. If we have a class of functions for which the series is convergent and unique then this set of functions will form an infinite dimensional vector space V. Every function in V can be expressed as a linear combination of the functions $\cos nx$ and $\sin nx$. The functions $\cos nx$ and $\sin nx$ are orthonormal under the inner product

$$< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\,dx,$$

see the list of integrals at the beginning of Lecture 20. So the cosine and sine functions give an orthonormal basis for our vector space V. We will see in the Dirichlet theorem that V is a very large space of functions which includes all the functions with which we are reasonably familiar.

$\diamond$Theorem (Dirichlet): *Let $f = f(x)$ be a function given on $\pi < x < \pi$ and defined as periodic, of period 2π, on $\mathbb{R}$. Suppose*

1. *$f(x)$ is bounded.*
2. *$f(x)$ has a finite number of discontinuities on $(-\pi, \pi)$, and at each discontinuous point x_0, the two limits $\lim_{x\to x_0-0} f(x)$ and $\lim_{x\to x_0+0} f(x)$ exist.*

Then the Fourier series of f converges

$$\frac{1}{2}\Big(\lim_{x\to x_0-0} f(x) + \lim_{x\to x_0+0} f(x)\Big).$$

Expecially, the Fourier series of f converges to $f(x)$ at all points x where f is continuous.

We will not be giving a proof of this rather deep theorem.

Remarks:

- A function which fails one, or more, of the Dirichlet conditions may still have a Fourier series. For example $f(x) = \sin(\frac{1}{x})$, $x \in (-\pi, \pi)$ has a Fourier series but fails condition 2.

- Conditions 1, 2, 3 of the theorem are usually quite straightforward to check – we can usually draw a graph so 2 and 3 are easily verified. Condition 4 may be somewhat harder to check. However the large class of functions which are bounded on $(-\pi, \pi)$ will automatically satisfy this condition:

$$\begin{aligned} \text{if } |f(x)| &< M, \text{ a finite constant} \\ \text{then } \int_{-\pi}^{\pi} |f(x)|\,dx &< \int_{-\pi}^{\pi} M\,dx = 2\pi M < \infty. \end{aligned}$$

♦**Example** For the function below find the Fourier series and find the value of the series at any jump discontinuities.

$$f(x) = \begin{cases} \frac{\pi}{2}, & -\pi < x < -\frac{\pi}{2} \\ x + \frac{\pi}{2}, & -\frac{\pi}{2} < x < 0 \\ \frac{\pi}{2} - x, & 0 < x < \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} < x < \pi. \end{cases}$$

Here f is defined as a periodic function, period 2π, on the whole of $\mathbb{R}$.

Solution: First we calculate the Fourier coefficients.

$$\begin{aligned} a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} \frac{\pi}{2}\cos nx\,dx + \int_{-\frac{\pi}{2}}^{0} (x + \frac{\pi}{2})\cos nx\,dx + \int_{0}^{\frac{\pi}{2}} (\frac{\pi}{2} - x)\cos nx\,dx \right. \\ &\quad \left. + \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\cos nx\,dx\right] \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \frac{\pi}{2}\cos nx\,dx - 2\int_{0}^{\frac{\pi}{2}} x\cos nx\,dx\right], \end{aligned}$$

where in the second line we have collected together all the $\frac{\pi}{2}\cos nx$ integrands under the one integral. The remaining two integrals with integrands $x\cos nx$ can be combined into one. Why? There are the usual two cases to deal with. Firstly $n = 0$,

$$a_0 = \frac{1}{2}[x]_{-\pi}^{\pi} - \frac{1}{\pi}[x^2]_0^{\frac{\pi}{2}} = \frac{3\pi}{4}.$$

Secondly, $n \geq 1$

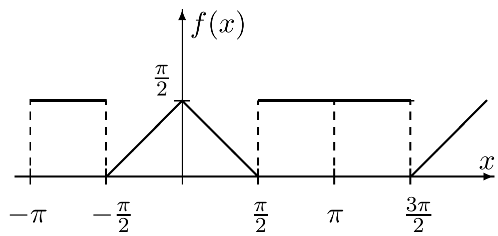
$$a_n = \frac{1}{2}\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\left(\frac{\sin nx}{n}\right)'\,dx$$

$$\begin{aligned}
&= -\frac{2}{\pi}\left[x\frac{\sin nx}{n}\right]_0^{\frac{\pi}{2}} + \frac{2}{\pi n}\int_0^{\frac{\pi}{2}} \sin nx\, dx \\
&= -\frac{1}{n}\sin(\frac{n\pi}{2}) + \frac{2}{\pi n^2}[1-\cos(\frac{n\pi}{2})] \\
&= \begin{cases} \frac{1}{2\pi n^2}[1-(-1)^m], & n=2m, \quad m=1,2,3,\ldots \\ \frac{(-1)^{m+1}}{2m+1} + \frac{2}{(2m+1)^2\pi}, & n=2m+1, \quad m=0,1,2,\ldots. \end{cases}
\end{aligned}$$

Now for the b_n. This time the integrals with integrands $x\sin nx$ cancel. Why? This leaves just one integral with integrand $\frac{\pi}{2}\sin nx$, we have

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-\pi}^{\pi} \sin nx\, dx \\
&= \frac{1}{2}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi} \\
&= 0.
\end{aligned}$$

$$f(x) = \begin{cases} \frac{\pi}{2}, & -\pi < x < -\frac{\pi}{2} \\ x+\frac{\pi}{2}, & -\frac{\pi}{2} < x < 0 \\ \frac{\pi}{2} - x, & 0 < x < \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} < x < \pi. \end{cases}$$



The Fourier series is

$$f(x) \sim \frac{3\pi}{8} + \left(\frac{2}{\pi} - 1\right)\cos x + \frac{1}{\pi}\cos 2x + \left(\frac{2}{9\pi} + \frac{1}{3}\right)\cos 3x + \cdots.$$

Or, in full generality

$$f(x) \sim \frac{3\pi}{8} + \sum_{m=0}^{\infty}\left[\frac{(-1)^{m+1}}{2m+1} + \frac{2}{(2m+1)^2\pi}\right]\cos[(2m+1)x] + \sum_{m=1}^{\infty}\frac{1}{2\pi m^2}[1-(-1)^m]\cos(2mx).$$

We must now check that f, as defined above, satisfies the four conditions of the Dirichlet theorem. We readily see from the graph that conditions 1, 2, and 3 are satisfied. Condition 4 is also satisfied; in fact, as $f \geq 0$ then condition 4 just says that the area under the curve $f = f(x)$ must be finite, the graph shows this is clearly true. The area is $\frac{3\pi^2}{4}$. We can now apply the Dirichlet theorem. We note that f is continuous on $(-\pi, \pi)$ except at the points $x = -\frac{\pi}{2}$ and $\frac{\pi}{2}$. At these two points of discontinuity the function jumps from 0 to $\frac{\pi}{2}$. According to the theorem the Fourier series will converge, at the jumps, to the mid-point of the jump. So at $x = \frac{-\pi}{2}$, and $\frac{\pi}{2}$ the series converges to $\frac{\pi}{4}$. At $x = \frac{\pi}{2}$ the series is

$$\frac{3\pi}{8} + \sum_{m=1}^{\infty}\frac{1}{2m^2\pi}[1-(-1)^m](-1)^m.$$

Re-arranging this expression

$$\frac{3\pi}{8} - \sum_{p=0}^{\infty} \frac{1}{\pi(2p+1)^2},$$

where we have used the fact that $(-1)^m - 1$ is zero for m even and -2 for $m = 2p+1$ odd. By the Dirichlet theorem this last expression must converge to $\frac{\pi}{4}$. From which we get the interesting result

$$\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} = \frac{\pi}{8}.$$

The following example is another demonstration of how the Dirichlet theorem can be used to obtain interesting infinite sums.

♦**Example** For the function $f(x) = e^x$, given on $-\pi < x < \pi$ and defined as periodic on $\mathbb{R}$, find the Fourier series and the value of that series at any jump discontinuities.

Solution: The integrals giving the Fourier coefficients are

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\, dx \quad \text{and} \quad \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\, dx.$$

These integrals are evaluated with two integrations by parts (using $v = e^x$ on each occasion). After the second integration you find that the remaining integral is proportional to the integral you started with, adding this back to the left side you can then obtain an algebraic expression for this integral. The results are

$$\begin{aligned} a_n &= \frac{1}{\pi}\left[e^x \frac{\cos nx \; + n \sin nx}{1+n^2}\right]_{-\pi}^{\pi} \\ &= \frac{(-1)^n}{\pi(1+n^2)}(e^{\pi} - e^{-\pi}) \\ b_n &= \frac{1}{\pi}\left[e^x \frac{\sin nx \; - n \cos nx}{1+n^2}\right]_{-\pi}^{\pi} \\ &= \frac{n(-1)^{1+n}}{\pi(1+n^2)}(e^{\pi} - e^{-\pi}). \end{aligned}$$

We have our Fourier series

$$e^x \sim \frac{1}{\pi}(e^{\pi} - e^{-\pi})\left\{\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \cos nx \; + \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{1+n^2} \sin nx\right\}.$$

It is a simple matter to verify that f satisfies all four Dirichlet conditions. We apply the theorem at the end-points, $x = \pi$ and $x = -\pi$. The mid-point of the jump in e^x

at the end-points is $\frac{1}{2}(e^{\pi}+e^{-\pi})$, according to the theorem this is the value to which the series must converge at $x=\pi$ (and $x=-\pi$). So we have

$$\begin{aligned} \frac{1}{2}(e^{\pi}+e^{-\pi}) &= \frac{1}{\pi}(e^{\pi}-e^{-\pi})\left\{\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\cos n\pi\right\} \\ &= \frac{1}{\pi}(e^{\pi}-e^{-\pi})\left\{\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{1+n^2}\right\}. \end{aligned}$$

Re-arranging this expression we have

$$\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{1+n^2}=\frac{\pi}{2}\left(\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}\right).$$

A highly non-trivial result.

25 Lecture 22

The Complex Form of the Fourier Series

The aim of this section is to exploit the very beautiful relationship between cosines, sines and exponentials – de Moivre's theorem. Recall,

$$\begin{aligned} e^{inx} &= \cos nx + i \sin nx \\ e^{-inx} &= \cos nx - i \sin nx \\ &\text{or} \\ \cos nx &= \frac{1}{2}(e^{inx} + e^{-inx}) \\ \sin nx &= \frac{1}{2i}(e^{inx} - e^{-inx}). \end{aligned}$$

Substitution of these expressions into our Fourier series will result in a sum of terms in e^{inx} and e^{-inx}. However, it is easier to obtain this *complex form* of the Fourier series directly. Suppose we have,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx = \sum_{n=-\infty}^{\infty} \frac{c_n}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx,$$

assuming convergence of the series. Now,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = \begin{cases} 1, & n = m \\ 0, & n \neq m. \end{cases}$$

Hence,

$$\sum_{n=-\infty}^{\infty} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = c_m.$$

Which determines the Fourier coefficients,

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx.$$

With our usual powerful hindsight we can now define the complex form of the Fourier series.

Suppose $f = f(x)$ is given on $-\pi < x < \pi$ and defined on $\mathbb{R}$ as a periodic function of period 2π. Then the complex Fourier series of f is

$$\diamond \quad f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\text{with } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

♦**Example** Obtain the complex form of the Fourier series of $f(x)$, periodic of period 2π on $\mathbb{R}$, where

$$f(x) = x, \quad -\pi < x < \pi.$$

Solution: The Fourier coefficients of the complex form of the Fourier series are defined as

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx.$$

There are two cases to consider: $n = 0$ and $n \neq 0$. Firstly, $n = 0$,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0.$$

For $n \neq 0$ we have

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} x\left(\frac{e^{-inx}}{-in}\right)'\,dx \\
&= \frac{1}{2\pi}\left\{\left[\frac{-xe^{-inx}}{in}\right]_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi} e^{-inx}\,dx\right\} \\
&= \frac{1}{2\pi}\left\{-\frac{2\pi}{in}(-1)^n + \left[\frac{1}{n^2}e^{-inx}\right]_{-\pi}^{\pi}\right\} \\
&= \frac{i}{n}(-1)^n.
\end{aligned}$$

Here we have used the fact that $e^{in\pi} + e^{-in\pi} = 2\cos n\pi = 2(-1)^n$. You should satisfy yourself that this is the same series found earlier. Hint: re-write the sum so that it is from $n = 1$ to $n = \infty$.

Other Intervals

Our Fourier series are based on the interval of length 2π, $(-\pi, \pi)$. However we can very easily adjust to intervals of any other length, $2L$ say, by a simple change of scale $x \longrightarrow \frac{\pi x}{L}$. Our interval will be $(-L, L)$. Or $(0, 2L)$ or whatever the application requires. The basis functions are

$$\cos(\frac{n\pi x}{L}) \text{ and } \sin(\frac{n\pi x}{L}) \text{ or } e^{\frac{in\pi x}{L}}.$$

For example, if our functions are given on $(-L, L)$ and defined to be periodic of period $2L$ we have

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{L}) + \sum_{n=1}^{\infty} b_n \cos(\frac{n\pi x}{L}) \\ \text{or } f(x) &\sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}. \end{aligned}$$

With Fourier coefficients given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx \\ c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi x}{L}}\,dx. \end{aligned}$$

For intervals of length $2L$ other than $(-L, L)$ the range of integration will need to be adjusted. For example, for $(0, 2L)$ the integrals will be $\int_0^{2L} \cdots$.

♦**Example** Re-work the Fourier series for $f(x) = x$ on the interval $(-L, L)$.

<u>Solution</u>: We leave the details as an exercise, the Fourier coefficients are

$$\begin{aligned} a_n &= 0 \\ b_n &= -\frac{2L}{n\pi}(-1)^n. \end{aligned}$$

26 Lecture 23

Odd and Even Functions

◇ <u>Definition</u>: Consider a function, $f = f(x)$, defined on a symmetric interval, $(-L, L)$.

- $f(x)$ is called an *even* function if $f(x) = f(-x)$.
- $f(x)$ is called an *odd* function if $f(x) = -f(-x)$.

The functions $\cos x$ and $\cos nx$ are even functions. The functions $\sin x$ and $\sin nx$ are odd. Most functions are neither odd nor even. However, all functions can be decomposed into a sum of an odd and an even function. In fact the Fourier series decomposes a function in this way, the even part is the cosine series and the odd part is the sine series. Generally we have $f(x) = f_e(x) + f_o(x)$, where $f_e(x) = \frac{1}{2}(f(x) + f(-x))$ is even and $f_o(x) = \frac{1}{2}(f(x) - f(-x))$ is odd. Even functions are reflection symmetric in the y axis. Odd functions are double reflection symmetric, reflect first in the y axis and then in the x axis.

If a function is even (or odd) then considerable simplification in the Fourier series results. The Fourier series of an even function consists only of cosine terms (including the $n = 0$ term). The Fourier series of an odd function consists only of sine terms. In fact we have

◇<u>Theorem</u>: *Suppose $f = f(x)$ is given on $(-L, L)$ and defined to be periodic, period $2L$ on $\mathbb{R}$.*

- *If f is an even function then*

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{(n\pi x)}{L}\,dx \\ b_n &= 0. \end{aligned}$$

  We call the resulting series a Fourier cosine *series.*

- *If f is an odd function then*

$$\begin{aligned} a_n &= 0 \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{(n\pi x)}{L}\,dx. \end{aligned}$$

  We call the resulting series a Fourier sine *series.*

Given a function on $(0, L)$ we can extend the function to the interval $(-L, L)$ in two natural ways.

◇ Definition: A function f defined on $(0, L)$ can be extended to $(-L, L)$ as

- the *even extension* by defining $f(x) = f(-x)$ for $x \in (-L, 0)$, so f is an even function on $(-L, L)$.

- the *odd extension* by defining $f(x) = -f(-x)$ for $x \in (-L, 0)$, so f is an odd function on $(-L, L)$.

The even extension will have a Fourier cosine series and the odd extension will have a Fourier sine series.

Suppose we are given a function $f = f(x)$ on an arbitrary interval, $(0, L)$, and asked to determine the Fourier series. There is no unique way in which this can be done. We might consider the function to be periodic on $\mathbb{R}$ of period L and develop the full Fourier series, with both sine and cosine terms. On the other hand we may extend f to be an odd (even) function on $(-L, L)$ by taking the odd (even) extension of f. The Fourier series will then consist entirely of sine (cosine) terms, a Fourier sine series (cosine series). Clearly a choice based on the context of the problem at hand has to be made.

♦**Example** Represent

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ -1, & 1 < x < 2, \end{cases}$$

as

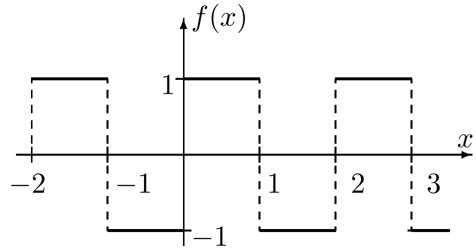
(a) a Fourier sine series

(b) a Fourier cosine series.

Solution: (a) We extend f to a periodic (period 4 on $\mathbb{R}$) odd function and calculate the Fourier sine series according to the theorem above.

Our function is now an odd function on $(-2, 2)$ with $L = 2$. So we have

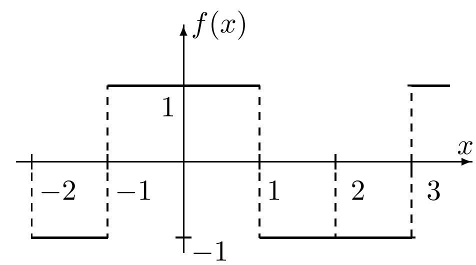
$$f(x) = \begin{cases} 1, & -2 < x < -1 \\ -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \\ -1, & 1 < x < 2. \end{cases}$$



$$\begin{aligned} a_n &= 0 \\ b_n &= \int_0^1 \sin(\frac{n\pi x}{2})\,dx - \int_1^2 \sin(\frac{n\pi x}{2})\,dx \\ &= \left[-\frac{2\cos(\frac{n\pi x}{2})}{n\pi}\right]_0^1 - \left[-\frac{2\cos(\frac{n\pi x}{2})}{n\pi}\right]_1^2 \\ &= \begin{cases} \frac{2}{m\pi}[1-(-1)^m], & n = 2m \\ 0, & n = 2m+1. \end{cases} \end{aligned}$$

(b) We extend f to an even function on $(-2, 2)$ with $L = 2$ and calculate according to our theorem

$$f(x) = \begin{cases} -1, & -2 < x < -1 \\ 1, & -1 < x < 0 \\ 1, & 0 < x < 1 \\ -1, & 1 < x < 2. \end{cases}$$



$$\begin{aligned} a_n &= \int_0^1 \cos(\frac{n\pi x}{2})\,dx - \int_1^2 \cos(\frac{n\pi x}{2})\,dx \\ &= \left[\frac{2\sin(\frac{n\pi x}{2})}{n\pi}\right]_0^1 - \left[\frac{2\sin(\frac{n\pi x}{2})}{n\pi}\right]_1^2 \\ &= \frac{4}{n\pi}\sin(\frac{n\pi}{2}) \\ &= \begin{cases} 0, & n = 2m \\ \frac{4}{m\pi}(-1)^m, & n = 2m+1. \end{cases} \end{aligned}$$

We have used the identity $\sin(\frac{(2m+1)\pi}{2}) = (-1)^m$.

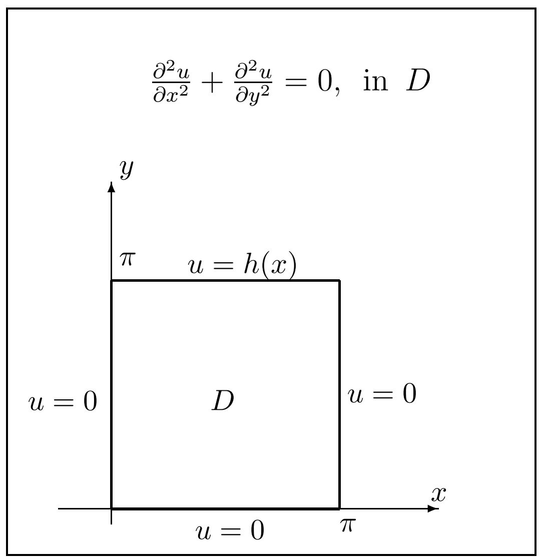
27 Lecture 24

An Application of Fourier Series

The fact that almost any function can be written uniquely as a Fourier series turns out to have great utility. One application that exploits this fact is the solving of boundary value problems – including those of fluid dynamics. In this lecture we will consider a simple boundary value problem for Laplace's equation in two dimensions.

Let u be a function given as harmonic (i.e. satisfies Laplace's equation) on the interior of the square, $D = \{(x, y) : \ 0 < x < \pi, \ \ 0 < y < \pi\}$. We must prescribe boundary conditions for u on the four sides of the square. We take, $u(0, y) = 0$, $u(\pi, y) = 0$, $u(x, 0) = 0$ and $u(x, \pi) = h(x)$, where h is a given function.

We will attack this problem by first finding a solution which satisfies the homogeneous boundary conditions. That is the three boundary conditions with zero right hand sides. We then take a sum of all such solutions in an attempt to get a Fourier series which will satisfy the inhomogeneous boundary condition, $u(x, \pi) = h(x)$.



We start by assuming a separable solution

$$u = X(x)Y(y).$$

Substituting this into Laplace's equation we get, after dividing by $u = XY$,

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0,$$

or,

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}.$$

Generally we cannot have "a function of x" equals "a function of y", unless each is constant. We take

$$\frac{X''(x)}{X(x)} = -\lambda.$$

We have to decide on the sign of λ. If its positive X will be a linear combination of sines and cosines. If λ is negative then we have a linear combination of exponentials for X. However, the two boundary conditions $u(0, y) = u(\pi, y) = 0$ will require $X(0) = X(\pi) = 0$. Only the trigonometric functions can satisfy both conditions. So we take $\lambda = \beta^2$ and we have

$$X'' + \beta^2 X = 0.$$

This is the simple harmonic motion equation, with general solution

$$X = A\cos\beta x + B\sin\beta x \;\; \text{for} \;\; \beta \neq 0,$$
$$X = C_1 x + C_2, \;\; \text{for} \;\; \beta = 0.$$

Applying the boundary conditions $X(0) = X(\pi) = 0$ we find that the case with $\beta = 0$ is trivial as we must have $C_1 = C_2 = 0$. Applying these two boundary conditions to the first case we have, for an non-trivial solution, $A = 0$ and $\sin\beta\pi = 0$, so that $\beta = n$, $n = 1, 2, 3, \ldots$. So X is

$$X = B\sin nx.$$

Now return to the equation for Y. Remember we have $\lambda = \beta^2 = n^2$,

$$Y'' - n^2 Y = 0,$$

a second order linear ordinary differential equation. This type of equation is solved by first solving the characteristic equation (recall, MATH 101). In the present case the solutions of the characteristic equation are n and $-n$. The solution is

$$Y = C_3 e^{ny} + C_4 e^{-ny}.$$

Apply the boundary condition at $y = 0$, we require $u(x, 0) = 0$ which, for our separable solution, means $Y(0) = 0$. So that $C_3 + C_4 = 0$ and Y takes the form

$$Y = C_3(e^{ny} - e^{-ny}).$$

Our solution is $u = XY$, in fact have one solution for each $n = 0, 1, 2, \ldots$. We denote these solutions as u_n, they are

$$u_n = B_n(e^{ny} - e^{-ny})\sin nx,$$

where we have combined the constants B and C_3 into the new constant B_n. The u_n satisfy Laplace's equation and all three homogeneous boundary conditions – i.e.

$u = 0$ for $x = 0$, $x = \pi$ and $y = 0$. Clearly, u_n cannot satisfy the final boundary condition, $u(x, \pi) = h(x)$, for a general function h. However, because Laplace's equation is linear any sum of the u_n will also be a solution. Summing the u_n gives a Fourier sine series for y fixed $(y = \pi)$. All we need do is to choose the B_n so that this sine series is the correct series for the odd extension of $h(x)$. We have

$$h(x) = u(x, \pi) = \sum_{n=1}^{\infty} u_n(x, \pi) = \sum_{n=1}^{\infty} B_n(e^{n\pi} - e^{-n\pi}) \sin nx.$$

So $B_n(e^{n\pi} - e^{-n\pi})$ corresponds to our usual Fourier coefficients b_n. This is the Fourier series of an odd extension so our coefficients are determined as

$$B_n(e^{n\pi} - e^{-n\pi}) = \frac{2}{\pi} \int_0^{\pi} h(x) \sin nx \, dx,$$

or,

$$B_n = \frac{2}{\pi(e^{n\pi} - e^{-n\pi})} \int_0^{\pi} h(x) \sin nx \, dx.$$

With the determination of the B_n the problem is completely solved. The solution,

$$u = \sum_{n=1}^{\infty} B_n(e^{ny} - e^{-ny}) \sin nx$$

satisfies Laplace's equation and all four boundary conditions.

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Question 1 [20 marks]

Let (in the cylindrical polar coordinate)

$$\mathbf{F} = 2\rho z \sin\phi \hat{\rho} + \rho z \cos\phi \hat{\phi} + \rho^2 \sin\phi \hat{\mathbf{k}}.$$

(a) Find the divergence and the curl of $\mathbf{F}$.

(b) Prove that the vector field $\mathbf{F}$ is conservative, and then find an appropriate potential.

Question 2 [15 marks]

Use Gauss' divergence theorem to calculate

$$\int\int_{\partial\Omega} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS,$$

where

$$\mathbf{F} = (x + e^y)\mathbf{i} - e^{z^2}\mathbf{j} + z^2\mathbf{k},$$

$$\Omega = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\},$$

and $\hat{\mathbf{n}} = \hat{\mathbf{n}}(x, y, z)$ is the outward pointing unit normal vector of $\partial\Omega$ at (x, y, z).

Question 3 [15 marks]

Verify Stokes' theorem by computing $\int_{\partial S} \mathbf{F} \cdot d\,\mathbf{r}$ and $\int\int_S (curl\mathbf{F}) \cdot \hat{\mathbf{n}} \, dS$, where

$$\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k},$$

S is the portion of the surface $z^2 = x^2 + y^2$ below the plane $z = 1$ and above the plane $z = 0$.

Question 4 [20 marks]

Let $\mathbf{v}$, μ and p be the velocity vector, the density and the pressure of the fluid respectively. Then $\mathbf{v}$, μ and p satisfy the continuity equation:

$$\frac{\partial \mu}{\partial t} + \nabla \cdot (\mu \mathbf{v}) = 0, \tag{1}$$

and the Euler's equation:

$$\frac{\partial \mathbf{v}}{\partial t} + \omega \times \mathbf{v} = \mathbf{F} - \frac{1}{\mu}\nabla p - \nabla(\frac{1}{2}v^2), \tag{2}$$

where $v^2 = \mathbf{v} \cdot \mathbf{v}$, $\omega = curl\mathbf{v}$, and $\mathbf{F}$ is the body force per unit mass.

(a) Simplify the continuity equation and the Euler's equation, if the flow is incompressible and is steady, that is, μ is constant, $\mathbf{v}$ and p do not explicitly depend on t.

(b) Suppose a large volume of incompressible fluid rotates steadily about a fixed vertical axis $\mathbf{k}$, with velocity vector $\mathbf{v} = \rho^2\hat{\phi}$ in the cylindrical coordinates. Suppose further the density of the fluid $\mu = \mu_0$, and the body force per unit mass $\mathbf{F} = -g\mathbf{k}$, where μ_0 and g are positive constants.

   (i) Show that $\mathbf{v} = \rho^2\hat{\phi}$ satisfies the continuity equation.

   (ii) Use the Euler's equation to find the pressure p.

Question 5 [20 marks]

Let $f(x) = |x|$, $-\pi < x < \pi$.

(a) Find the Fourier series of $f(x)$.

(b) By choosing suitable x in the interval $(-\pi, \pi)$, prove

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Question 6 [10 marks]

Suppose that $Y(y)$ satisfies

$$Y''(y) - \beta^2 Y(y) = 0, \quad 0 < y < \pi,$$

and $Y(0) = Y(\pi) = 0$, where $\beta > 0$ is a constant. Show that $Y(y) = 0$, $0 < y < \pi$.