I extracted some real questions from BB and I put the answers here.

1. On sets
2. On matrices
3. Mathematical shortcuts
4. The polynomial function
5. The exponential function
6. Practical induction
7. Absolute value
8. Boolean minimisation
The difference between \{a, b\} and (a, b)

- The former represents a set; the order of elements doesn’t matter
- The latter represents an ordered pair; the order of elements does matter
- An ordered pair is an element of the cartesian product of two sets
- If \( A = \{a, b\} \), then the fact that
  - the element \( a \) belongs to set \( A \) is written \( a \in A \)
  - the subset of \( A \) containing only the element \( a \) is written \( \{a\} \subseteq A \), or \( \{a\} \subseteq \{a, b\} \)
- The power set of a set is the set containing all subsets of the given set as elements:
  - if \( A = \{a, b, c\} \) then \( \mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \)
- The number of elements in a set is called the cardinality of the set.
  - If a set \( A \) has \( n \) elements, then \( \mathcal{P}(A) \) has \( 2^n \) elements.
- The subset relation, \( \subseteq \), is a partial order relation on the powerset, \( \mathcal{P}(A) \), of a set \( A \). This means it is reflexive, antisymmetric, and transitive.
The partition of a set

- A collection of nonempty sets \( \{A_1, \cdots, A_n\} \) is a **partition** of a set \( A \) iff
  - \( A = A_1 \cup A_2 \cup \cdots \cup A_n \),
  - \( A_i \cap A_j = \emptyset, \forall i \neq j \in \{1, 2, \ldots, n\} \), that is \( A_1, \cdots, A_n \) are mutually disjoint.

- Of course, the sets \( A_1, A_2, \cdots, A_n \) must be included in the set \( A \), so they must be elements of the power set of \( A \).

- Any equivalence relation defined on a set \( A \) is a part (subset) of the cartesian product \( A \times A \) which is reflexive, symmetric, and transitive.

- Any equivalence relation defined on a set \( A \) creates a partition on \( A \), where the elements (sets) of the partition are the equivalence classes of the relation.
The difference of two sets

- Given two sets $A$ and $B$, the difference of $A$ minus $B$, written $A \setminus B$ or $A - B$ is the set $A - B = \{x | x \in A \text{ and } x \notin B\}$.

- From the above definition, it is clear that $A - B \neq B - A$. In other words, the set difference is not commutative.

- Given an universal set $U$ (all other sets of interest are just subsets of it), then the difference $U - A$ is called the complement of $A$ and is denoted by $A'$. We note that some texts use $\overline{A}$ or $\mathbb{C}A$ instead of $A'$ to denote the complement of the set $A$. Hence

  $$\mathbb{C}A = \overline{A} = A' = U - A = U \setminus A = \{x \in U | x \notin A\}.$$  

- De Morgan laws apply to the complement

  $$\mathbb{C}(A \cup B) = \mathbb{C}A \cap \mathbb{C}B$$

  $$\mathbb{C}(A \cap B) = \mathbb{C}A \cup \mathbb{C}B$$
Matrix Multiplication

- We multiply matrices “row by column”:
- Let $2 \times 2$ (i.e. 2 by 2) matrices $A$ and $B$ be given respectively by

$$A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 4 \\ 1 & -2 \end{bmatrix}$$

Find $AB$ and $A^2$.

- To find the element $a_{11}$ of the result,
  - we multiply each element from the first row in the first matrix by the corresponding element on the first column in the second matrix and then
  - we sum up these products.

- We proceed in a similar way for all the other elements of the result, meaning that for the element $a_{ij}$ in the resulting matrix we will consider row number $i$ in the first matrix and column number $j$ in the second matrix.
\[ AB = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 7 & 4 \\ 1 & -2 \end{bmatrix} \]

\[ = \begin{bmatrix} 2 \times 7 + (-1) \times 1 & 2 \times 4 + (-1) \times (-2) \\ 5 \times 7 + 3 \times 1 & 5 \times 4 + 3 \times (-2) \end{bmatrix} = \begin{bmatrix} 13 & 10 \\ 38 & 14 \end{bmatrix}, \]

\[ A^2 = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \]

\[ = \begin{bmatrix} 2 \times 2 + (-1) \times 5 & 2 \times (-1) + (-1) \times 3 \\ 5 \times 2 + 3 \times 5 & 5 \times (-1) + 3 \times 3 \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ 25 & 4 \end{bmatrix} \]

- As expected, \( A^2 = A \times A, \ A^3 = A^2 \times A, \ldots, \ A^{n+1} = A^n \times A \)
Let $3 \times 3$ matrices $A$ and $B$ be given by

$$A = \begin{bmatrix} 2 & -1 & 13 \\ 5 & 3 & -6 \\ 11 & 0 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 4 & -8 \\ 1 & -2 & 0 \\ 9 & -4 & -3 \end{bmatrix}.$$ 

Then the matrix product $AB$ is

$$AB = \begin{bmatrix} 2 & -1 & 13 \\ 5 & 3 & -6 \\ 11 & 0 & 10 \end{bmatrix} \times \begin{bmatrix} 7 & 4 & -8 \\ 1 & -2 & 0 \\ 9 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 2 \times 7 + (-1) \times 1 + 13 \times 9 & 2 \times 4 + (-1) \times (-2) + 13 \times (-4) & 2 \times (-8) + (-1) \times 0 + 13 \times (-3) \\ 5 \times 7 + 3 \times 1 + (-6) \times 9 & 5 \times 4 + 3 \times (-2) + (-6) \times (-4) & 5 \times (-8) + 3 \times 0 + (-6) \times (-3) \\ 11 \times 7 + 0 \times 1 + 10 \times 9 & 11 \times 4 + 0 \times (-2) + 10 \times (-4) & 11 \times (-8) + 0 \times 0 + 10 \times (-3) \end{bmatrix} = \begin{bmatrix} 130 & -42 & -55 \\ -16 & 38 & -22 \\ 167 & 4 & -118 \end{bmatrix}.$$
Mathematical shortcuts

- The **sigma notation** uses the Greek letter Σ (Sigma) to denote the sum of the terms of a sequence in a condensed form, by specifying the start and the end values of an index, that is

\[ a_1 + a_2 + \cdots + a_m = \sum_{j=1}^{m} a_j \]

- The **big-pi notation** uses the Greek letter Π (Pi) to denote the product of the terms of a sequence in condensed form, by specifying the start and the end values of an index, that is

\[ a_1 \times a_2 \times \cdots \times a_m = \prod_{j=1}^{m} a_j \]

- **iff** is a shortcut for *if and only if*

- **P iff Q** means that *if P then Q AND if Q then P.*
The polynomial function $f(x) = x^n$

- The base is the variable and the power is the fixed number.
- The standard form of a polynomial is
  \[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 \]
- The $a_i \in \mathbb{R}$ are called coefficients.
- The term $a_n$ is assumed to be non-zero and is called the leading term.
- The degree of the polynomial is the largest exponent of $x$ which appears in the polynomial.
- A polynomial with one term is called a monomial.
- A degree 0 polynomial is a constant.
- A degree 1 polynomial is a linear function, a degree 2 polynomial is a quadratic function, a degree 3 polynomial is a cubic function, etc.
- The behavior of a polynomial function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$ is essentially the same as that of the monomial $a_n x^n$ over a large enough scale.
The exponential function $f(x) = a^x$

- The base is the fixed number, and the power is the variable
- The base:
  - always some positive number other than 1
  - if less than 1, the function is decreasing
  - if greater than 1, the function is increasing
- Some rules to manipulate exponential functions:
  
  $$a^{x+y} = a^x \cdot a^y \quad a^{xy} = (a^x)^y \quad a^0 = 1 \quad a^{-x} = \frac{1}{a^x}$$

- Exponential growth is bigger and faster than polynomial growth.
- The exponential function $e^x$ having base $e = 2.718\ldots$ is very important in mathematics.
Practical induction

Follow these two steps:

1. **Check.** Take the smallest possible value(s) for $n$ and see if the given relation is true.
2. **Inductive step.** Assume the given relation is true for a value, say $n$, and prove that it is also true for $n + 1$.

Conclude the relation is true for any $n$. 
For any real number $a$, its **absolute value** (or **modulus**) is the numerical value of $a$ without regard to its sign and is denoted by $|a|$, that is:

$$|a| = \begin{cases} 
  a & \text{if } a \geq 0 \\
  -a & \text{if } a < 0 
\end{cases}$$

Because the square-root of a positive number represents the positive square root (without a sign), it follows that

$$|a| = \sqrt{a^2}$$

can sometimes be used as a definition of absolute value.

In a similar way, for any complex number $z = x + iy$, where $x$ and $y$ are real numbers and $i = \sqrt{-1}$, the **absolute value** or **modulus** of $z$ is denoted $|z|$ and is defined as

$$|z| = \sqrt{x^2 + y^2}$$
Given the numbers $a, b, c$, the absolute value has the following six fundamental properties:

1. $|a| \geq 0$
2. $|a| = 0 \iff a = 0$ or $|a - b| = 0 \iff a = b$
3. $|ab| = |a||b|
4. $\left| \frac{a}{b} \right| = \frac{|a|}{|b|}$
5. $|a| \leq c \iff -c \leq a \leq c$
6. $|a| - |b| \leq |a + b| \leq |a| + |b|

The last one is called triangle inequality.
Boolean minimisation

- The keys to Boolean minimisation lie in the following theorems:

\[
\begin{align*}
    a + ab &= a & a \cdot (a + b) &= a \\
    a + a'b &= a + b & a \cdot (a' + b) &= a \cdot b \\
    a \cdot b + a \cdot b' &= a & (a + b) \cdot (a + b') &= a
\end{align*}
\]

- The last one is the key to Karnaugh maps too.