# AMTH140 DISCRETE MATHEMATICS
# RECURRENCE RELATIONS

You may recall from primary school questions like

What is the next number in
3, 6, 12, ...
or
1, 1, 2, 3, 5, ....?

The first one is straight forward, 24. The second is much harder, 8. The first one is just doubling the previous term and can be written as

$$a_{n+1} = 2a_n.$$

This description is not strictly correct because the sequence depends on what the first term is, a different first term would produce a different sequence. The correct description is

$$a_{n+1} = 2a_n, \ a_0 = 3.$$

It is a tradition in this area of mathematics to have the lowest subscription as $n$ with $n$ starting at $n = 0$..

The second sequence is the famous Fibonacci sequence where each term, after the second, is the sum of the previous 2 terms. This can be written

$$a_{n+2} = a_{n+1} + a_n, \ a_0 = 1, \ a_1 = 1.$$

If the first 2 terms are different, there would be a different sequence.

The first term(s) is(are) called the initial value(s). In some cases they are not supplied and so only a very general description can be made.

These two examples are examples of recurrence relations. You met another example in Tutorial 1. The first one is called first order because the gap between the subscripts is 1. The second example is called second order because the gap between the largest and smallest subscripts is 2. They are both linear recurrence relations because there is NO multiplication of terms, multiplication by $n$ and so on.

The plan is to find a way to solve this type of recurrence relation with emphasis on the second order ones. By "solve" I mean find a formula for $a_n$, the general term, in terms of just $n$.

The first step is to find the solution to

$$a_{n+1} = 2a_n, \ a_0 = 3$$

by finding $a_1, a_2, a_3$ then use the pattern shown to find a formula. Then show that that formula satisfies the recurrence relation.

The second step is to use this information to obtain a more efficient method then the third step is to apply these ideas to a second order linear recurrence relation.

Back to the first example

$$\begin{aligned}
n = 0 \qquad & a_1 = 2a_0 = 2 \times 3 \\
n = 1 \qquad & a_2 = 2a_1 = 2 \times 2 \times 3 = 2^2 \times 3 \\
n = 2 \qquad & a_3 = 2a_2 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3.
\end{aligned}$$

Notice how the 2's keep 'piling up'. Also notice that the power of 2 is the same as $a$'s subscript. So the conjecture is

$$a_n = 2^n \times 3.$$

Now to confirm that this is correct

$$\begin{aligned}
a_{n+1} &= 2^{n+1} \times 3 = 2 \times 2^n \times 3 = 2 \times a_n \\
a_0 &= 2^0 \times 3 = 3.
\end{aligned}$$

Correct.

So the role of the coefficient of $a_n$ is crucial.

Here is the method that was worked out, many years ago, after a lot of trial and error.

1. Bring all the $a$'s across to the LHS.

$$a_{n+1} - 2a_n = 0.$$

2. Use the coefficients, 1 and $-2$, to write an equation in $x$ (or $t$ or $y$ or ....).

$$x - 2 = 0.$$

This is called the characteristic equation.

3. Solve for $x$.

$$x = 2.$$

4. Put $a_n = A \times 2^n$ where $A$ is some constant to be found by using the initial condition.

$$n = 0,\ a_0 = 3 \ \text{ so } \ 3 = A \times 2^0 = A$$

$$\therefore\ a_n = 3 \times 2^n.$$

If there are no initial conditions just leave it as

$$a_n = A \times 2^n.$$

Now for a second order linear recurrence relation (try saying that 5 times quickly)

Consider $a_{n+2} = 3a_{n+1} - 2a_n, \qquad a_0 = 1,\ a_1 = 2.$

Follow the same steps as above.

$$a_{n+1} - 3a_{n+1} + 2a_n = 0$$

Characteristic equation: $x^2 - 3x + 2 = 0$.

The gap of 2 between the largest and smallest subscripts produces the $x^2$. A third order equation would produce a cubic equation for its characteristic equation.

$$\text{Solve:} \qquad \begin{aligned} (x-2)(x-1) &= 0 \\ x &= 2,\ 1. \end{aligned}$$

(Think of this as two versions of the example above

$$x - 2 = 0 \qquad \text{and} \qquad x - 1 = 0$$

or

$$a_{n+1} = 2a_n \qquad \text{and} \qquad a_{n+1} = a_n.)$$

$$
\begin{aligned}
a_n &= A \times 2^n + B \times 1^n \\
&= A2^n + B
\end{aligned}
$$

$$
\begin{aligned}
n = 0 \qquad\qquad 1 &= A \times 1 + B \\
n = 1 \qquad\qquad 2 &= A \times 2 + B
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ A = 1, &\qquad B = 0 \\
\therefore\ a_n &= 2^n.
\end{aligned}
$$

If you have time it is always a good idea to substitute your answer back into the original recurrence relation, as a check.

Fairly straight forward? There are two possible complications

(a) When the characteristic equation has a repeated root, $(x-3)^2 = 0$ for example.

(b) When the RHS at step 1 is not zero.

When the RHS is zero, the equation is called homogeneous. So the example just above is a second order linear homogeneous recurrence relation. (Try saying that three times, quickly.)

When the RHS is not zero the equation is called nonhomogeneous. I'll tackle that problem shortly, now back to Problem 1.

**Example** Solve $a_{n+2} - 6a_{n+1} + 9a_n = 0$.

$$
\begin{aligned}
x^2 - 6x + 9 &= 0 \\
(x-3)^2 &= 0 \\
x &= 3, \quad \text{multiplicity 2.}
\end{aligned}
$$

It is no use trying $a_n = A \times 3^n + B \times 3^n$ because that is $a_n = (A+B) \times 3^n$, only one $3^n$.

To be able to incorporate 2 occurrences of $3^n$ try throwing in an extra $n$.

$$
\begin{aligned}
a_n &= An3^n + B3^n \\
&= (An + B)3^n.
\end{aligned}
$$

You check to see if it is a solution.

If the root has multiplicity 3, say $(x-2)^3 = 0$ then to obtain 3 occurrences of $2^n$ try

$$\begin{aligned} a_n &= An^2 2^n + Bn2^n + C2^n \\ &= (An^2 + Bn + C)2^n. \end{aligned}$$

Notice the pattern? The degree of the polynomial in the brackets is one less then the multiplicity of the root.

Next Problem 2, the nonhomogeneous case.

Many years ago it was discovered that if you pretend that if it is a homogeneous one, and solve it, then that solution will 'work'! But the solution is only part of the answer. Problem 2 now becomes two sub-problems.

Problem 2(a) when the RHS is not like any of the expressions in the homogeneous solution.

Problem 2(b) when the RHS is like one of the expressions in the homogeneous solution.

Before tackling these two problems, some notation. Use $u_n$ for the solution to the homogeneous case and $v_n$ for the other part of the solution. Then the final solution is $a_n = u_n + v_n$. Why? When $a_n$ is substituted into the original recurrence relation, the $u_n$ part produces zero and the $v_n$ part produces the RHS. So $a_n$ satisfies the equation.

Once $a_n$ has been found, use any initial condition to get rid of any constants.

Problem 2(a). Working on the basis that what is substituted into the LHS must come out as the RHS, try $v_n$ = constant $\times$ the general form of the RHS.

For example, if the RHS is $n^2$, try $u_n = An^2 + Bn + C$. Substitute this into the original equation and solve for $A, B$ and $C$.

**Example** Solve $a_{n+2} - 3a_{n+1} + 2a_n = 3^n,\ n \geq 0$.

$$\begin{aligned} \text{Characteristic equation:} \qquad x^2 - 3x + 2 &= 0 \\ (x-2)(x-1) &= 0 \\ x &= 2, 1 \end{aligned}$$

$$\begin{aligned} u_n &= A2^n + B1^n \\ &= A2^n + B \quad \text{as before.} \end{aligned}$$

Now substitute $v_n = C3^n$ into the original equation.

$$C3^{n+2} - 3C3^{n+1} + 2C3^n = 3^n.$$

Divide throughout by $3^n$.

$$\begin{aligned} C3^2 - 3C3 + 2C &= 1 \\ 2C &= 1 \\ C &= \frac{1}{2} \\ \therefore\ v_n &= \frac{1}{2} \times 3^n \\ \therefore\ a_n = u_n + v_n &= A2^n + B + \frac{3^n}{2}. \end{aligned}$$

Problem 2(b). Here the general form of the RHS cannot be used, as it is part of $u_n$. This problem splits into 2 sub-problems.

Problem 2(b) (i). The characteristic equation has distinct roots.

Problem 2(b) (ii). The characteristic equation has repeated roots.

If the characteristic equation has distinct roots, the solution is a variation on the answer to Problem 2(a). Here, throw in an extra $n$. For example, if the characteristic equation has a root of $x = 3$ and the $RHS$ is $3^n$, use $v_n = Cn3^n$. That is, substitute $v_n = C_n 3^n$ into the original equation and find a value for $C$.

The solution to Problem 2(b) (ii) is a variation on the solution to Problem 2(b)(i). In fact, they are compatible. If the multiplicity is 2, throw in an $n^2$. If the multiplicity is 3, throw in a $n^3$; and so on. Do you see how the solutions to Problem 2(b)(i) and 2(b)(ii) are compatible?

For example, if the characteristic equation has a root of $x = 5$ of multiplicity 2 and the RHS is $5^n$, try $v_n = Cn^2 5^n$.

Now have a look at the examples in Chapters 21 to 24 and the Supplementary Exercises. Also read through the Summary on pages 161-163.