Question 1 [10 marks]

(a) Suppose $A$ is the set of distinct letters in the word *elephant*, $B$ is the set of distinct letters in the word *sycophant*, $C$ is the set of distinct letters in the word *fantastic*, and $D$ is the set of distinct letters in the word *student*. The universe $U$ is the set of 26 lower-case letters of the English alphabet. Find

(i) $A \cup B$
(ii) $A \cap C$
(iii) $A \cap (C \cup D)$
(iv) $(A \cup B \cup C \cup D)'$

(b) Find two finite sets $A$ and $B$ such that $A \in B$ and $A \subset B$.

(c) Give a proof of or a counterexample to the following statement:

$$A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$$

Answer

(a) The sets $A$, $B$, $C$, $D$, and $U$ are (lower case letters!):

(0.5 marks) $A = \{e, l, p, h, a, n, t\}$
(0.5 marks) $B = \{s, y, c, o, p, h, a, n, t\}$
(0.5 marks) $C = \{f, a, n, t, s, i, c\}$
(0.5 marks) $D = \{s, t, u, d, e, n\}$
(0.5 marks) $U = \{a, b, c \ldots, x, y, z\}$

Then

(i) (1 mark) $A \cup B = \{e, l, p, h, a, n, t, s, y, c, o\}$
(ii) (1 mark) $A \cap C = \{a, n, t\}$
(iii) (1 mark) $A \cap (C \cup D) = \{e, a, n, t\}$ where

$$C \cup D = \{f, a, n, t, s, i, c, u, d, e\}$$

(iv) (1.5): (1 marks) $(A \cup B \cup C \cup D)' = \{b, g, j, k, m, q, r, v, w, x, z\}$ because

$$(+ 0.5 \text{ marks}) (A \cup B \cup C \cup D) = \{e, l, p, h, a, n, t, s, y, c, o, f, i, u, d\}$$

(b) (2 marks) For example, $A = \{x\}$, $B = \{x, \{x\}\}$

(c) (1 mark) Counterexample: let $A = \{a\}$, $B = \{b\}$, $C = \{b\}$, then

$$A \cup B = \{a, b\}, A \cup C = \{a, b\}, B \cup C = \{b\},$$
$$A \cap (B \cup C) = \emptyset \text{ but } (A \cup B) \cap (A \cup C) = \{a, b\}$$
Question 2 [6 marks]

Prove by mathematical induction that, for all \( n \geq 1 \),

\[
1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}
\]

Answer

Let \( S_n \) denote the above statement. Then

(a) **(2 marks)** \( S_1 \) is true because \( 1 = \frac{1(3 \cdot 1 - 1)}{2} = \frac{2}{2} = 1 \).

(b) **(4 marks)** We assume \( S_n \) is true, that is

\[
1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}
\]

If we add \( 3(n + 1) - 2 = 3n + 1 \) to both sides of this equality, we obtain

\[
1 + 4 + 7 + \cdots + (3n - 2) + (3n + 1) = \frac{n(3n - 1)}{2} + (3n + 1)
\]

\[
= \frac{n(3n - 1) + 6n + 2}{2}
\]

\[
= \frac{3n^2 + 5n + 2}{2}
\]

\[
= \frac{(n + 1)(3(n + 1) - 1)}{2}
\]

i.e., the statement \( S_{n+1} \) is true, which completes the proof by induction.

Question 3 [6 marks]

Given the function \( f : \mathbb{N} \to \mathbb{R}, f(x) = 2x^4 + x^3 + 3 \) prove from definition that \( f(x) \in \Theta(x^4) \).

Answer

**(3 marks)** Let \( g(x) = x^4 \). Observe that \( \forall x > 0, x^3 + 3 > 0 \). Then \( 2x^4 \leq 2x^4 + x^3 + 3 \) and because all terms on the both sides of the inequality are positive, \( 2|x^4| \leq |2x^4 + x^3 + 3| \), that is \( D|g(x)| \leq |f(x)| \), where \( D = 2 \).

**(3 marks)** For \( x > 1, 1 \leq x^4 \) and \( x^3 < x^4 \), so \( 2x^4 + x^3 + 3 \leq 2x^4 + x^4 + 3x^4 \leq 6x^4 \). Because all terms on the both sides of the inequality are positive, \( |2x^4 + x^3 + 3| \leq 6|x^4| \), that is \( |f(x)| \leq C|g(x)| \), where \( C = 6 \) and \( M = 1 \).

In other words, \( 2|x^4| \leq |2x^4 + x^3 + 3| \leq 6|x^4| \), that is \( f(x) \in \Theta(x^4) \).
Question 4

(a) Given the proposition

\[ p : \text{John is tall and John is thin} \]

write down its negation \( \sim p \). (Hint: use De Morgan’s laws).

(b) Given a proposition \( p \rightarrow q \) we know that neither its converse \( (q \rightarrow p) \), nor its inverse \((\sim p \rightarrow \sim q)\) is a consequence of it. Using the truth table method prove that \( p \rightarrow q \) and its contrapositive are logically equivalent.

(c) The contrapositive proof technique uses the above identity: if we want to prove \( p \rightarrow q \), it is exactly the same thing to prove its contrapositive. Use the contrapositive proof technique to prove that

if the square of an integer \( m \) is even, then the integer \( m \) must be even.

Answer

(a) (3 marks) \( p \) is actually \( a \land b \), where

\[ a : \text{John is tall}, \]
\[ b : \text{John is thin}. \]

It means that \( \sim p = \sim (a \land b) = \sim a \lor \sim b \), i.e.,

\[ \sim p : \text{John is not tall or John is not thin}. \]

(b) (2 marks) The contrapositive of \( p \rightarrow q \) is \( \sim q \rightarrow \sim p \). (0.5 marks) for this.

\[
\begin{array}{ccc|ccc}
 p & q & \sim p & \sim q & p \rightarrow q & \sim q \rightarrow \sim p \\
 F & F & T & T & T & T \\
 F & T & T & F & T & T \\
 T & F & F & T & F & F \\
 T & T & F & F & T & T \\
\end{array}
\]

The last two columns are identical hence the sentence and its contrapositive are equivalent.

(c) (3 marks) We must prove that if \( m^2 \) is even, then the integer \( m \) must be even, that is \( p \rightarrow q \), where \( [p : m^2 \text{ is even}] \) and \([q : m \text{ is even}]\). The contrapositive of this statement is

\( \sim q \rightarrow \sim p \), that is if \( m \) is not even, then \( m^2 \) is not even (1 mark).

In other words, if \( m \) is odd, then \( m^2 \) is odd.

Now, (2 marks) if \( m \) is odd, then \( m = 2k + 1 \) for some integer \( k \).

Then \( m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 \) which is an odd number.
Question 5

(a) Rewrite the following formal statement in a variety (at least two) of equivalent but more informal ways (plain English). Do not use the symbols $\forall$ and $\exists$.

$$\forall x \in \mathbb{R}, x^2 \geq 0$$

(b) Rewrite the following statement formally. Use quantifiers, variables and predicates.

There is a barber who shaves all men in town who do not shave themselves.

(c) A universal conditional statement is written formally as

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$$

A universal conditional statement is not logically equivalent to its converse.

(i) Write the preceding statement in symbols.

(ii) Give an example.

Answer

(a) (2 marks) Any two or more from the set:

(i) all real numbers have non-negative squares
(ii) every real number has a non-negative square
(iii) any real number has a non-negative square
(iv) $x$ has a non-negative square, for each real number $x$
(v) the square of any real number is non-negative

(b) (2 marks) $\exists \text{ barber}(x) \land \forall \text{ man}(y) \land \neg \text{shaves}(y, y) \rightarrow \text{shaves}(x, y)$

(c) (i) (2 marks) $\forall x \in D, \text{ if } P(x) \text{ then } Q(x) \neq \forall x \in D, \text{ if } Q(x) \text{ then } P(x)$

(ii) (2 marks) $\forall x \in \mathbb{R}, \text{ if } x > 3 \text{ then } x^2 > 9$ has the converse $\forall x \in \mathbb{R}, \text{ if } x^2 > 9 \text{ then } x > 3$

Observe that the sentence is true whereas its converse is false, since (for instance) $(-4)^2 = 16 > 9$ but $-4 \neq 3$
Question 6

Construct a table showing the interchanges that occur at each step when selection sort is applied to the following list:

5, 3, 4, 6, 2

Answer

<table>
<thead>
<tr>
<th>steps</th>
<th>lists</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5 3 4 6 2</td>
</tr>
<tr>
<td>1</td>
<td>2 3 4 6 5</td>
</tr>
<tr>
<td>2</td>
<td>2 3 4 6 5</td>
</tr>
<tr>
<td>3</td>
<td>2 3 4 6 5</td>
</tr>
<tr>
<td>4</td>
<td>2 3 4 5 6</td>
</tr>
<tr>
<td>5</td>
<td>2 3 4 5 6</td>
</tr>
</tbody>
</table>

(4 marks) Must show all these steps and no others.

(4 marks) Explain everything.

Any other solution: (1 mark).

Question 7

Consider the following graph:

Figure 1: Your first graph

(a) Write the edge set, the vertex set, and give a table showing the edge–endpoints function \( \sigma \).

(b) Find all edges that are incident on \( v_1 \), all vertices that are adjacent to \( v_3 \), all edges that are adjacent to \( e_4 \), all loops, all parallel edges, all vertices that are adjacent to themselves, and all isolated vertices.

(c) Show that the graph bellow does (not) have an Euler circuit.
Figure 2: Your second graph

Answer

(a) (0.5 marks) $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$,
(0.5 marks) $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$
(1 mark) the edge $\mapsto$ endpoints set function is given by

<table>
<thead>
<tr>
<th>edge</th>
<th>edgepoints set $\sigma(e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>${v_1, v_2}$</td>
</tr>
<tr>
<td>$e_2$</td>
<td>${v_1, v_3}$</td>
</tr>
<tr>
<td>$e_3$</td>
<td>${v_1, v_3}$</td>
</tr>
<tr>
<td>$e_4$</td>
<td>${v_2, v_3}$</td>
</tr>
<tr>
<td>$e_5$</td>
<td>${v_5, v_6}$</td>
</tr>
<tr>
<td>$e_6$</td>
<td>${v_5}$</td>
</tr>
<tr>
<td>$e_7$</td>
<td>${v_6}$</td>
</tr>
</tbody>
</table>

(b) (1 mark) $\{e_1, e_2, e_3\}$ are incident to $v_1$
(1 mark) $\{v_1, v_2\}$ are adjacent to $v_3$
(1 mark) $\{e_1, e_2, e_3\}$ are adjacent edges to edge $e_4$
(1 mark) $e_6$ and $e_7$ are loops
(1 mark) $e_2$ and $e_3$ are parallel
(1 mark) $v_5$ and $v_6$ are adjacent to themselves
(1 mark) $v_4$ is an isolated vertex

(c) (1 mark) Theorem: if a graph has an Euler circuit, then every vertex of the graph has even degree.
Contrapositive: If some vertex of a graph has an odd degree, then the graph does not have an Euler circuit.
Vertices $v_1$ and $v_3$ both have degree 3, which is an odd number. Therefore, this graph does not have an Euler circuit.

Question 8 [9 marks]
(a) Draw a binary tree to represent the following mathematical expression:

\[(a + b) \ast (c + d)\]

(b) Write the prefix (preorder traversal) and postfix (postorder traversal) forms of the expression (of the binary tree).

(c) Use Kruskal's algorithm to find a minimal spanning tree for the following graph, where the numbers represent the weight of the corresponding edges. What is the total weight of the minimal spanning tree?

Figure 3: Another graph

Answer

(a) (3 marks) The binary tree:

Figure 4: Binary tree
(b) (3 marks)

prefix: \( * + a \ b + c \ d \)
postfix: \( a \ b + c \ d + * \)

(c) (3=2+1 marks) Order of adding edges (from Kruskal’s algorithm table):

\( \{a, b\}, \{e, f\}, \{e, d\}, \{d, c\}, \{g, f\}, \{b, c\} \)

![Figure 5: Spanning tree](image)

(1 mark) The total weight of the minimal spanning tree is 23.

Question 9 [9 marks]

Given the number \( X = 123_x \), where \( x \) is the smallest possible base for it, convert \( X \) in bases 2, 8, 10 and hexa.

Answer

If \( X = 123_x \), then the smallest possible base for it is four, so \( X = 123_4 \). (5 marks)

It means that \( X = 123_4 = 1 \times 4^2 + 2 \times 4^1 + 3 \times 4^0 = 27_{10} \) (1 marks)

But \( 27_{10} = 16 + 8 + 2 + 1 = 2^4 + 2^3 + 2^1 + 2^0 = 11011_2 \). (1 marks)

To get the number in base 8 = 23, we make groups of three digits from right to left:

\( 11011_2 = 11|011 = 33_8 \). (1 marks)

To get the number in base 16 = 24, we make groups of four digits from right to left:

\( 11011_2 = 1|1011 = 1B_{16} \). (1 marks)

Question 10 [10 marks]

Let \( S \) be a partition of the set \( X = \{1, 2, 3, 4, 5, 6\} \) consisting of three subsets, each of them containing at most three elements of the initial set. Define \( x \ R \ y \) to mean that for some set \( S \in S \), both \( x \) and \( y \) belong to \( S \).

(a) Give \( R \) explicitly by its elements.
(b) Prove that $R$ is an equivalence relation.

(c) Draw the digraph of the relation $R$.

Answer

(a) \(4 = 2 + 2 \text{ marks}\) Because $X$ has six elements and the three sets in the partition can have at most three elements,

\[6 = 3 + 2 + 1 = 2 + 2 + 2\]

we distinguish two possible ways to partition the set $X$, containing, respectively, 3, 2, 1 or 2, 2, 2 distinct elements.

(2 marks) Let us assume $S_1 = \{\{1, 3, 5\}, \{2, 6\}, \{4\}\}$. Then, the complete relation is

$R_1 = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5), (2, 2), (2, 6), (6, 2), (6, 6), (4, 4)\}$

(2 marks) Let us assume $S_2 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. Then, the complete relation is

$R_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5), (5, 6), (6, 5), (6, 6)\}$

(b) \(3 = 1 + 1 + 1 \text{ marks}\) Let $x \in X$. Then, by the definition of partition, $x$ belongs to some member $S$ of $S$. Thus $xRx$ and $R$ is reflexive. (1 mark)

Suppose that $xRy$. Then both $x$ and $y$ belong to some set $S \in S$. Since both $y$ and $x$ belong to $S$, $yRx$ and $R$ is symmetric. (1 mark)

Suppose that $xRy$ and $yRz$. Then both $x$ and $y$ belong to some set $S \in S$ and both $y$ and $z$ belong to some set $T \in S$. Since $y$ belongs to exactly one member of $S$, we must have $S = T$. Therefore, both $x$ and $z$ belong to $S$ and $xRz$. We have shown that $R$ is transitive. (1 mark)

(c) \(3 \text{ marks}\) The digraph of the relation $R_1$ is shown below:

![Figure 6: The digraph of relation R](image-url)
The digraph of the relation $R_2$ is the picture in the middle repeated three times. We can see that $R$ is

- reflexive: there is a loop at every vertex
- symmetric: for every directed edge from $a$ to $b$, there is also a directed edge from $b$ to $a$
- transitive: if there is a directed edge from $a$ to $b$ and a directed edge from $b$ to $c$, then there is a directed edge from $a$ to $c$

Question 11

(a) Draw a gate implementation for a One-Bit Equality Circuit: the output of this circuit is 1 if and only if both inputs are 0 or both inputs are 1.

(b) Find the canonical form for $f = xy + z'$.

(c) Explicitly define the canonical form for $f = xy + z'$ by means of a truth table.

Answer

(a) (3 marks) One-Bit Equality Circuit:

Figure 7: One-Bit Equality Circuit
(b) (3 marks) The computation:

\[ f = xy + z' \]
\[ = xy(z + z') + (x + x')(y + y')z' \]
\[ = xyz + xyz' + xyz' + xy'z' + x'y'z' \]
\[ = xyz + (xyz' + xyz') + xy'z' + x'y'z' \]
\[ = xyz + xyz' + xy'z' + x'y'z' \]

(c) (3 marks) The truth table for the canonical form is:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>f(x,y,z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>1</td>
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</tbody>
</table>

Question 12 [7 marks]

Let us assume that \( a \) and \( b \) are given constants and that the two initial values \( s_0 \) and \( s_1 \) are known for the recurrence relation

\[ s_n = as_{n-1} + bs_{n-2} \]

(a) Find the general solution for the case where \( b = 0 \).

(b) Find the general solution for the case where \( a = 0 \).

(c) Give examples for both cases.

Answer

The two cases are easy to deal with.

(a) (2 marks) Let \( b = 0 \), so that \( s_n = as_{n-1} \) for \( n \geq 1 \). Then \( s_1 = as_0 \), \( s_2 = as_1 = a^2s_0 \), etc. A simple induction argument shows that \( s_n = a^n s_0 \) for all \( n \in \mathbb{N} \).

(b) (2 marks) Let \( a = 0 \), then \( s_n = bs_{n-2} \). Then \( s_2 = bs_0 \), \( s_4 = bs_2 = b^2s_0 \), etc. , so that \( s_{2n} = b^n s_0 \) for all \( n \in \mathbb{N} \). Similarly, \( s_3 = bs_1 \), \( s_5 = bs_3 = b^2s_1 \), etc. , so that \( s_{2n+1} = b^n s_1 \) for all \( n \in \mathbb{N} \).

(c) (3 marks) Consider the recurrence relation \( s_n = 3s_{n-1} \) with \( s_0 = 5 \). Here \( a = 3 \) and so \( s_n = 5 \cdot 3^n \) for all \( n \in \mathbb{N} \).

Consider the recurrence relation \( s_n = 3s_{n-2} \) with \( s_0 = 5 \) and \( s_1 = 2 \). Here \( b = 3 \) and so \( s_{2n} = 5 \cdot 3^n \) and \( s_{2n+1} = 2 \cdot 3^n \) for all \( n \in \mathbb{N} \).
Please remember – This examination question paper MUST BE HANDED IN. Failure to do so may result in the cancellation of all marks for this examination. Writing your name and number on the front will help us confirm that your paper has been returned.