AMTH140 EXAMINATION 2002

SOLUTIONS

Question 1

(a) If $A = \{a,b,c\}$ and $B = \{d,e\}$ find

(i) $\mathcal{P}(A)$, the power set of $A$

(ii) $A \times B$.

(b) If $C = \{1,2,3\}$ does $\mathcal{P}(C)$, the power set of $C$, form a partition of $C$? Give reasons for your answer.

(c) Use the set identity $X - Y = X \cap Y'$ to prove

$$(A - B) \cap (C - B) = (A \cap C) - B.$$  

[10 marks]

Question 2

Prove by mathematical induction that $n \leq 3^n$ for $n \in \mathbb{N}$.  

[5 marks]

Question 3

Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be defined by

$$f(n) = \frac{n^2 + \log_2 n}{n + 1}$$

Prove from the definition that $f(n) = O(n)$.  

[6 marks]

Question 4

(a) Write down a truth table to show that $(\sim (p \lor q))$ is equivalent to $(\sim p) \land (\sim q)$.

(b) Show that the argument

$p \lor (\sim q) \rightarrow \sim r, (\sim s) \lor q, t, (\sim p) \rightarrow (\sim t), p \land (\sim r) \rightarrow s, \ldots q$

is valid by deducing the conclusion from the premises step by step through the use of the basic rules of inference or laws of logic.  

[8 marks]
Question 5

(a) Translate the following statement into a sentence in English.

\[ \forall m, \ n \in \mathbb{N} \ ( (m \div n) \in \mathbb{N} \vee (n \div m) \in \mathbb{N}) . \]

(b) What is the negation of the following statement?

\[ \forall x \in \mathbb{R}, \ \exists y \in \mathbb{R}, \ ((x = y^2) \lor (x < 0)) . \]

Simplify your expression. [9 marks]

Question 6

Use the Quicksort algorithm to sort alphabetically the following list.

\[ P, \ M, \ Q, \ T, \ K. \]

Use the first element as a pivot, underline the pivot elements and use an asterick (*) to mark the elements in their final positions. How many comparisons are needed in this case? [5 marks]

Question 7

(a) A complete graph \( K_n \) has \( n \) vertices and each pair of vertices are joined by an edge.

For which values of \( n \) is the graph \( K_n \) planar? Justify your answer.

(b) For the following graph

\[ \begin{array}{ccc}
  v_1 & - & v_2 \\
  & - & v_3 \\
\end{array} \]

(i) Give the adjacency matrix for the graph.

(ii) How many walks from vertex \( v_1 \) to \( v_2 \) are there of length 3?

(iii) Is the graph Eulerian? Justify your answer. [10 marks]
Question 8

(a) Draw a binary tree to represent the following mathematical expression

\[ \frac{(a - b)}{(c \times (d - e))} \]

(b) Write down the vertex sequence for the postorder traversal of the tree in (a).

(c) Find a minimal spanning tree for the following weighted graph where the numbers represent the weight of the corresponding edge. What is the total weight of the minimal spanning tree?

![Graph with vertices and edges labeled with weights]

[11 marks]

Question 9

Perform the following arithmetic operations in the bases indicated

(a) 11101 × 111 \ (base 2)

(b) F7 - B6 \ (base 16).

Write out each of these in base ten notation. \ [8 marks] 

Question 10

(a) Let \( A = \{0, 1, 2\} \) and \( R \) be a binary relation on \( \mathcal{P}(A) \) defined by

\[ (B, C) \in R \ \text{iff} \ B \subseteq C. \]

That is, \( R \) is the “subset relation”, \( \subseteq \), on \( \mathcal{P}(A) \).

Question 10 (a)(i) is on page 5.
Question 10 continued

(i) Show that $R$ is reflexive, antisymmetric and transitive. That is, show that $R$ is a partial order relation.

(ii) Draw the corresponding Hasse diagram for the relation $R$.

(iii) Give the least element and the greatest element, if they exist.

(b) Let $V$ denote the set of vertices of the following graph $G$.

\begin{center}
\begin{tikzpicture}
    \node[vertex] (v1) at (0,0) {}; \node[vertex] (v2) at (1,0) {}; \node[vertex] (v3) at (2,0) {}; \node[vertex] (v4) at (1,1) {}; \node[vertex] (v5) at (2,1) {};
    \path[thick,->] (v1) edge (v2) edge (v3) edge (v4) edge (v5);
    \path[thick,->] (v2) edge (v3) edge (v4);
    \path[thick,->] (v3) edge (v4) edge (v5);
    \path[thick,->] (v4) edge (v5);
\end{tikzpicture}
\end{center}

Define a relation $R$ on $V$ by

$(v, w) \in R$ iff there exists at least one edge in $G$
which connects vertex $v$ to vertex $w$ directly.

Is $R$ an equivalence relation? Justify your answer. \hspace{1cm} [10 marks]

Question 11

(a) Simplify the following switching circuit by writing down the corresponding Boolean expression, simplifying it then drawing the simplified circuit.

\begin{center}
\begin{tikzpicture}
    \node[vertex] (a) at (0,0) {}; \node[vertex] (b) at (0,-1) {}; \node[vertex] (d) at (1,-1) {a'};
    \path[thick,->] (a) edge (b) edge (a'); \path[thick,->] (b) edge (d);
\end{tikzpicture}
\end{center}
Question 11 continued

(b) The table below specifies a Boolean function \( f : S \times S \times S \rightarrow S \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
<th>( f(x, y, z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Give a Boolean expression corresponding to this function.

(c) Give the logic gate implementation of \( xy + x' \). \([8\text{ marks}]\)

Question 12

(a) Find the solution of the recurrence relation

\[
a_{n+2} - a_{n+1} - 2a_n = 0, \quad n \geq 0
\]

satisfying the initial conditions

\( a_0 = 3 \) and \( a_1 = 3 \).

(b) Find the general solution of the recurrence relation

\[
a_{n+2} - 5a_{n+1} + 6a_n = 2^n, \quad n \geq 0.
\]

\([10\text{ marks}]\)