Solution of Linear Nonhomogeneous Recurrence Relations

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A Case for Thought

- We already mentioned that finding a particular solution for a nonhomogeneous problem can be more involved than those exemplified in the previous lecture.
- Let us first highlight our point with the following example.

Example

Solve $a_{n+2} + a_{n+1} - 6a_n = 2^n$ for $n \geq 0$.

Solution.

- First we observe that the homogeneous problem
  \[ u_{n+2} + u_{n+1} - 6u_n = 0 \]

  has the general solution $u_n = A2^n + B(-3)^n$ for $n \geq 0$ because the associated characteristic equation $\lambda^2 + \lambda - 6 = 0$ has 2 distinct roots $\lambda_1 = 2$ and $\lambda_2 = -3$.
- Since the r.h.s. of the nonhomogeneous recurrence relation is $2^n$, if we formally follow the strategy in the previous lecture, we would try $v_n = C2^n$ for a particular solution.
- But there is a difficulty: $C2^n$ fits into the format of $u_n$ which is a solution of the homogeneous problem.
  - In other words, it can’t be a particular solution of the nonhomogeneous problem.
A Case for Thought

- This is really because 2 happens to be one of the two roots $\lambda_1$ and $\lambda_2$.
- However, we suspect that a particular solution would still have to have $2^n$ as a factor, so we try $v_n = Cn2^n$.
- Substituting it to $v_{n+2} + v_{n+1} - 6v_n = 2^n$, we obtain

$$C(n+2)2^{n+2} + C(n+1)2^{n+1} - 6Cn2^n = 2^n,$$

i.e., $10C2^n = 2^n$ or $C = \frac{1}{10}$.
- Hence a particular solution is $v_n = \frac{n}{10}2^n$ and the general solution of our nonhomogeneous recurrence relation is

$$a_n = A2^n + B(-3)^n + \frac{n}{10}2^n, \quad n \geq 0.$$

- In general, it is important that a correct form, often termed ansatz in physics, for a particular solution is used before we fix up the unknown constants in the solution ansatz. The following theorem can help.
Theorem (The Rational Roots Test.)

Let \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) be a polynomial with real coefficients and \( a_n \neq 0 \). If \( P(x) \) has rational roots, they are of the form \( \pm \frac{p}{q} \) where \( p \mid a_0 \) and \( q \mid a_n \).

- The choice of the form of a particular solution, covering the cases in this current lecture as well as the previous one, can be summarized below.
Method of Undetermined Coefficients

Consider a linear, constant coefficient recurrence relation of the form

\[ c_m a_{n+m} + \cdots + c_1 a_{n+1} + c_0 a_n = g(n) , \quad c_0 c_m \neq 0 , \quad n \geq 0 . \quad (*) \]

Suppose function \( g(n) \), the nonhomogeneous part of the recurrence relation, is of the following form

\[ g(n) = \mu^n (b_0 + b_1 n + \cdots + b_k n^k) , \quad (**) \]

where \( k \in \mathbb{N} \), \( \mu, b_0, \cdots, b_k \) are constants, and \( \mu \) is a root of multiplicity \( M \) of the associated characteristic equation

\[ c_m \lambda^m + \cdots + c_1 \lambda + c_0 = 0 . \]

Then a particular solution \( v_n \) of (*) should be sought in the form

\[ v_n = \left[ \sum_{i=0}^{k} B_i n^i \right] \mu^n n^M = \mu^n (B_0 + B_1 n + \cdots + B_{k-1} n^{k-1} + B_k n^k) n^M \quad (***) \]

where constants \( B_0, \cdots, B_k \) are to be determined from the requirement that \( a_n = v_n \) should satisfy the recurrence relation (*).
Method of Undetermined Coefficients

\[ v_n = \mu^n (B_0 + B_1 n + \cdots + B_{k-1} n^{k-1} + B_k n^k) n^M \] \hspace{1cm} (***)

- Obviously, the \( v_n \) in (*** \) is composed of two parts:
  - \[ \mu^n (B_0 + B_1 n + \cdots + B_k n^k) \] (which is of the same form of \( g(n) \) in (**) )
  - \( n^M \), which is a necessary adjustment for the case when \( \mu \), appearing in \( g(n) \) in (**), is also a root (of multiplicity \( M \)) of the characteristic equation of the associated homogeneous recurrence relation.

- If \( \mu \) is not a root of the characteristic equation, then just choose \( M = 0 \), implying alternatively that \( \mu \) is a "root" of 0 multiplicity.
We can also try \( \tilde{v}_n = \mu^n \left( \sum_{j=0}^{k+M} A_j n^j \right) \).

- If we rewrite \( \tilde{v}_n \) as \( \mu^n \sum_{j=M}^{k+M} A_j n^j + \mu^n \sum_{j=0}^{M-1} A_j n^j \), then
  - the first part is essentially (**}), while
  - the second part is just a solution of the homogeneous problem.
- It is however obvious that \( v_n \) in (**}) is simpler than \( \tilde{v}_n \).
We briefly hint why $v_n$ is chosen in the form of (***)

Let $\Delta$, $f(\lambda)$ and $P_s(\lambda)$ be defined in the same way as we did in the derivation hints of the theorem in the lecture just before the previous one, and we’ll also make use of some intermediate results there.

Recall that (*) can be written as $f(\Delta)a_n = g(n)$ and (**) implies $g(n) \in P_k(\mu)$.

1. If $f(\mu) \neq 0$, then $f(\Delta) P_k(\mu) \subseteq P_k(\mu)$. Hence if we try 
   $v_n = (B_0 + B_1 n + \cdots + B_k n^k)\mu^n \in P_k(\mu)$, then we can derive a set of 
   exactly $(k + 1)$ linear equations in $B_0, \cdots, B_k$, which can be used to 
   determine these $B_i$’s.

2. If $\mu$ is a root of $f(\lambda) = 0$ with multiplicity $M \geq 1$, then 
   $$f(\Delta) P_{M-1}(\mu) \subseteq \{0\} , \quad f(\Delta) P_{M+k}(\mu) \subseteq P_k(\mu) .$$

Hence if we try $v_n = n^M (B_0 + \cdots + B_k n^k)\mu^n \in P_{M+k}(\mu)$, we’ll again have a set of exactly $(k + 1)$ linear equations as the coefficients of the terms 
$\mu^n, \mu^n n, \cdots, \mu^n n^k$. The $(k + 1)$ constants $B_0, \cdots, B_k$ can thus be 
determined from these linear equations.
Examples

Example

Find the general solution of \( f(n+2) - 6f(n+1) + 9f(n) = 5 \times 3^n, \; n \geq 0. \)

Solution. Let \( f(n) = u_n + v_n, \) with \( u_n \) being the general solution of the homogeneous problem and \( v_n \) a particular solution.

(a) Find \( u_n: \) The associated characteristic equation \( \lambda^2 - 6\lambda + 9 = 0 \) has a repeated root \( \lambda = 3 \) with multiplicity 2. Hence the general solution of the homogeneous problem \( u_{n+2} - 6u_{n+1} + 9u_n = 0, \; n \geq 0 \) is \( u_n = (A + Bn)3^n. \)

(b) Find \( v_n: \) Since the r.h.s. of the recurrence relation, the nonhomogeneous part, is \( 5 \times 3^n \) and 3 is a root of multiplicity 2 of the characteristic equation (i.e. \( \mu = 3, \; k = 0, \; M = 2 \)), we try due to (***)

\[
v_n = B_0 \mu^n \times n^M \equiv Cn^23^n: \text{ we just need to observe that } C3^n \text{ is of the form } 5 \times 3^n \text{ and that the extra factor } n^2 \text{ is due to } \mu = 3 \text{ being a double root of the characteristic equation. Thus}
\]

\[
5 \times 3^n = v_{n+2} - 6v_{n+1} + 9v_n = C(n + 2)^23^{n+2} - 6C(n + 1)^23^{n+1} + 9Cn^23^n \\
= 18C3^n. \text{ Hence } C = \frac{5}{18} \text{ and } v_n = \frac{5}{18}n^23^n. \text{ The general solution is}
\]

\[
f(n) = \left(A + Bn + \frac{5}{18}n^2\right)3^n, \; n \geq 0.
\]
Example

Find the particular solution of
\[ a_{n+4} - 5a_{n+3} + 9a_{n+2} - 7a_{n+1} + 2a_n = 3, \quad n \geq 0 \]
satisfying the initial conditions \( a_0 = 2, \ a_1 = -\frac{1}{2}, \ a_2 = -5, \ a_3 = -\frac{31}{2} \).

Solution.

- We first find the general solution \( u_n \) for the homogeneous problem.
- We then find a particular solution \( v_n \) for the nonhomogeneous problem without considering the initial conditions.
- Then \( a_n = u_n + v_n \) would be the general solution of the nonhomogeneous problem.
- We finally make use of the initial conditions to determine the arbitrary constants in the general solution so as to arrive at our required particular solution.
Examples

Example

Find the particular solution of
\[ a_{n+4} - 5a_{n+3} + 9a_{n+2} - 7a_{n+1} + 2a_n = 3, \quad n \geq 0 \]
satisfying the initial conditions \( a_0 = 2, \ a_1 = -\frac{1}{2}, \ a_2 = -5, \ a_3 = -\frac{31}{2}. \)

Solution.

(a) Find \( u_n \): Since the associated characteristic equation \( \lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2 = 0 \) has the sum of all the coefficients being zero, i.e. \( 1 - 5 + 9 - 7 + 2 = 0 \), it must have a root \( \lambda = 1 \). After factorising out \( (\lambda - 1) \) via \( \lambda^4 - 5\lambda^3 + 9\lambda^2 - 7\lambda + 2 = (\lambda - 1)(\lambda^3 - 4\lambda^2 + 5\lambda - 2) \), the rest of the roots will come from \( \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0 \). Notice that \( \lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0 \) can again be factorised by a factor \( (\lambda - 1) \) because \( 1 - 4 + 5 - 2 = 0 \). This way we can derive in the end that the roots are

\[ \lambda_1 = 1 \text{ with multiplicity } m_1 = 3, \text{ and } \]
\[ \lambda_2 = 2 \text{ with multiplicity } m_2 = 1. \]

Thus the general solutions for the homogeneous problem is

\[ u_n = (A + Bn + Cn^2)1^n + D2^n, \]
or simply \( u_n = A + Bn + Cn^2 + D2^n \) because \( 1^n \equiv 1. \)
Examples

Example

Find the particular solution of \(a_{n+4} - 5a_{n+3} + 9a_{n+2} - 7a_{n+1} + 2a_n = 3\), \(n \geq 0\) satisfying the initial conditions \(a_0 = 2\), \(a_1 = -\frac{1}{2}\), \(a_2 = -5\), \(a_3 = -\frac{31}{2}\).

Solution.

(b) Find \(v_n\): Notice that the nonhomogeneous part is a constant 3 which can be written as \(3 \times 1^n\) when cast into the form of (**) and that 1 is in fact a root of multiplicity 3. In other words, we have in (***) \(\mu = 1\), \(k = 0\) and \(M = 3\). Hence we try a particular solution \(v_n = En^3 \cdot 1^n = En^3\). The substitution of \(v_n\) into the nonhomogeneous recurrence equations then gives

\[
3 = v_{n+4} - 5v_{n+3} + 9v_{n+2} - 7v_{n+1} + 2v_n
\]

\[
= E(n + 4)^3 - 5E(n + 3)^3 + 9E(n + 2)^3 - 7E(n + 1)^3 + 2En^3
\]

\[
= E(n^3 + 3n^2 \times 4 + 3n \times 4^2 + 4^3) - 5E(n^3 + 3n^2 \times 3 + 3n \times 3^2 + 3^3)
\]

\[
+ 9E(n^3 + 3n^2 \times 2 + 3n \times 2^2 + 2^3) - 7E(n^3 + 3n^2 \times 1 + 3n \times 1^2 + 1^3) + 2En^3
\]

\[
= -6E
\]

i.e., \(E = -\frac{1}{2}\). Hence \(v_n = -\frac{n^3}{2}\).
Example
Find the particular solution of \( a_{n+4} - 5a_{n+3} + 9a_{n+2} - 7a_{n+1} + 2a_n = 3 \), \( n \geq 0 \) satisfying the initial conditions \( a_0 = 2 \), \( a_1 = -\frac{1}{2} \), \( a_2 = -5 \), \( a_3 = -\frac{31}{2} \).

Solution.

(c) The general solution of the nonhomogeneous problem is thus
\[
a_n = u_n + v_n = A + Bn + Cn^2 + D2^n - \frac{n^3}{2}.
\]

(d) We now ask the solution in (c) to comply with the initial conditions.

<table>
<thead>
<tr>
<th>Initial Conditions</th>
<th>Induced Equations</th>
<th>Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_0 = 2 )</td>
<td>( A + D = 2 )</td>
<td>( A = 3 )</td>
</tr>
<tr>
<td>( a_1 = -1/2 )</td>
<td>( A + B + C + 2D = 0 )</td>
<td>( B = -2 )</td>
</tr>
<tr>
<td>( a_2 = -5 )</td>
<td>( A + 2B + 4C + 4D = -1 )</td>
<td>( C = 1 )</td>
</tr>
<tr>
<td>( a_3 = -31/2 )</td>
<td>( A + 3B + 9C + 8D = -2 )</td>
<td>( D = -1 )</td>
</tr>
</tbody>
</table>

Hence our required particular solution takes the following final form
\[
a_n = 3 - 2n + n^2 - \frac{n^3}{2} - 2^n, \quad n \geq 0.
\]
Examples

Example

Find the general solution of $a_{n+1} - a_n = n2^n + 1$ for $n \geq 0$.

Solution.

(a) The general solution for homogeneous problem is $u_n = A$ because the only root of the characteristic equation is $\lambda_1 = 1$.

(b) Since $n2^n + 1 = 2^n \times n + 1^n$ is of the form $\mu_1^n(b_1 n + b_0) + \mu_2^n c_0$ and $\mu_2 = 1$ is a simple root of the characteristic equation, we try the similar form $v_n = 2^n(B + Cn) + Dn$ for a particular solution. Substituting $v_n$ into the recurrence relation, we have

$$n2^n + 1 = v_{n+1} - v_n = 2^{n+1}(B + C(n + 1)) + D(n + 1) - 2^n(B + Cn) - Dn$$

$$= 2^n(Cn + B + 2C) + D \text{, i.e.,}$$

$$2^n [(C - 1)n + (B + 2C)] + (D - 1) = 0.$$ 

In order the above equation be identically 0 for all $n \geq 0$, we require all its coefficients to be 0, i.e., $C - 1 = 0$, $B + 2C = 0$, $D - 1 = 0$. Hence $B = -2$, $C = 1$ and $D = 1$ and the particular solution $v_n = 2^n(n - 2) + n$.

(c) The general solution is $u_n + v_n$ and thus reads

$$a_n = 2^n(n - 2) + n + A, \quad n \geq 0.$$
**Examples**

**Example**

Let $m \in \mathbb{N}$ and $S(n) = \sum_{i=0}^{n} i^m$ for $n \in \mathbb{N}$. Convert the problem of finding $S(n)$ to a problem of solving a recurrence relation.

**Solution.**

- We first observe

  $$S(n+1) = (n+1)^m + \sum_{i=0}^{n} i^m = S(n) + (n+1)^m.$$  

- Since the general solution will contain just 1 arbitrary constant, one initial condition should suffice.
- Hence $S(n)$ is the solution of

  $$S(n+1) - S(n) = (n+1)^m, \quad \forall n \in \mathbb{N}$$  

  $$S(0) = 0.$$