Solution of Linear Homogeneous Recurrence Relations
General Solutions for Homogeneous Problems

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If the characteristic equation associated with a given $m$-th order linear, constant coefficient, homogeneous recurrence relation has some repeated roots, then the solution given by $\sum A_i \lambda_i^n$ will not have $m$ arbitrary constants.
Introduction

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- To see this, we assume for instance $\lambda_1 = \lambda_2$, i.e. root $\lambda_1$ is repeated.
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- To see this, we assume for instance $\lambda_1 = \lambda_2$, i.e. root $\lambda_1$ is repeated.

- Then the solution $a_n = \sum_{i=1}^{m} A_i \lambda_i^n = (A_1 + A_2) \lambda_2^n + A_3 \lambda_3^n + \cdots + A_m \lambda_m^n$ has less than $m$ arbitrary constants because $A_1 + A_2$ comprises essentially only one arbitrary constant.
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To make up for the missing ones, we introduce the following more general theorem.
Theorem

Suppose the characteristic equation of the linear, constant coefficient recurrence relation

\[ c_m a_{n+m} + c_{m-1} a_{n+m-1} + \cdots + c_1 a_{n+1} + c_0 a_n = 0, \quad c_m c_0 \neq 0, \ n \geq 0 \]

has the following roots (all roots accounted for)

\[ \lambda_1, \ldots, \lambda_1, \ \lambda_2, \ldots, \lambda_2, \ \cdots, \lambda_k, \ldots, \lambda_k \]

such that \( \lambda_1, \ldots, \lambda_k \) are distinct, \( m_1, \ldots, m_k \geq 1 \) and \( m_1 + \cdots + m_k = m \).

Then the general solution of the recurrence relation is

\[
a_n = \sum_{i=1}^{k} \left[ \left( \sum_{j=0}^{m_i-1} A_{i,j} n^j \right) \lambda_i^n \right], \text{i.e.,}
\]

\[
a_n = (A_{1,0} + A_{1,1} n + \cdots + A_{1,m_1-1} n^{m_1-1}) \lambda_1^n + (A_{2,0} + A_{2,1} n + \cdots + A_{2,m_2-1} n^{m_2-1}) \lambda_2^n + \cdots + (A_{k,0} + A_{k,1} n + \cdots + A_{k,m_k-1} n^{m_k-1}) \lambda_k^n
\]

where \( A_{i,j}, \ \text{for} \ i = 1, \ldots, k \ \text{and} \ j = 0, \ldots, m_i - 1 \) are arbitrary constants.
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If the root was \( \lambda = 4 \) of multiplicity 3 then \( 4^n \) is multiplied by a polynomial of degree 2: \((An^2 + Bn + C)4^n\).
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- If the root was $\lambda = 4$ of multiplicity 3 then $4^n$ is multiplied by a polynomial of degree 2: $(An^2 + Bn + C)4^n$.

- If the root was $\lambda = 5$ of multiplicity 4 then $5^n$ is multiplied by a polynomial of degree 3: $(An^3 + Bn^2 + Cn + D)5^n$. 

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If the root was $\lambda = 5$ of multiplicity 4 then $5^n$ is multiplied by a polynomial of degree 3: $(An^3 + Bn^2 + Cn + D)5^n$.

etc.
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Find a particular solution of

\[ f(n + 2) + 4f(n + 1) + 4f(n) = 0, \quad n \geq 0 \]

with initial conditions \( f(0) = 1 \) and \( f(1) = 2 \).
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Solution. For clarity, we split the solution procedure into three steps below.

(a) The associated characteristic equation is \( \lambda^2 + 4\lambda + 4 = 0 \) which has a repeated root \( \lambda_1 = -2 \), i.e., \( m_1 = 2, \ k = 1 \).
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(b) The general solution from the theorem in this lecture is thus

\[ f(n) = (A_{1,0} + A_{1,1}n)\lambda_1^n = (B_0 + B_1n)(-2)^n, \]

where \( B_0 = A_{1,0} \) and \( B_1 = A_{1,1} \) are arbitrary constants.
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where \( B_0 = A_{1,0} \) and \( B_1 = A_{1,1} \) are arbitrary constants.

(c) Constants \( B_0 \) and \( B_1 \) are to be determined from the initial conditions

\[
\begin{align*}
  f(0) &= (B_0 + B_1 \times 0)(-2)^0 = 1 & \iff B_0 &= 1 \\
  f(1) &= (B_0 + B_1 \times 1)(-2)^1 = 2 & \implies -2(B_0 + B_1) &= 2.
\end{align*}
\]

They are thus \( B_0 = 1 \) and \( B_1 = -2 \). Hence the requested particular solution is

\[ f(n) = (1 - 2n)(-2)^n, \quad n \geq 0. \]
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- The associated characteristic equation is

\[ F(\lambda) \triangleq \lambda^3 - 3\lambda^2 + 4 = 0. \]
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  - We can check that \( \lambda_1 = -1 \) is indeed a root by verifying
    \[ F(-1) = (-1)^3 - 3(-1)^2 + 4 = 0. \]
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- To find the remaining roots, we first factorise \( F(\lambda) \) by performing a long division:
Example 2

\[
\begin{array}{ccc}
\lambda^2 & -4\lambda & +4 \\
\lambda^3 & -3\lambda^2 & +4 \\
\lambda^3 & +\lambda^2 & \\
-4\lambda^2 & 0 \\
-4\lambda^2 & -4\lambda & +4 \\
4\lambda & +4 & 0
\end{array}
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which implies \( \lambda^3 - 3\lambda^2 + 4 = (\lambda + 1)(\lambda^2 - 4\lambda + 4) = (\lambda + 1)(\lambda - 2)^2 \).
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Therefore all the roots, counting the corresponding multiplicity, are \(-1, 2, 2, \) i.e. \( \lambda_1 = -1, \ m_1 = 1 \) and \( \lambda_2 = 2, \ m_2 = 2. \)
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Therefore all the roots, counting the corresponding multiplicity, are \(-1, 2, 2,\) i.e. \(\lambda_1 = -1, m_1 = 1\) and \(\lambda_2 = 2, m_2 = 2.\)

Hence the general solution reads (with \(A = A_{1,0}, B = A_{2,0}, C = A_{2,1}\))

\[a_n = A(-1)^n + (B + Cn)2^n, \quad n \geq 0.\]
Example 3

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Find the particular solution of

\[ u_{n+3} + 3u_{n+2} + 3u_{n+1} + u_n = 0 \ , \ n \geq 0 \]

satisfying the initial conditions \( u_0 = 1 \), \( u_1 = 1 \) and \( u_2 = -7 \).
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Solution.

(a) The associated characteristic equation \( \lambda^3 + 3\lambda^2 + 3\lambda + 1 \equiv (\lambda + 1)^3 = 0 \) has roots \( \lambda_1 = -1 \) with multiplicity \( m_1 = 3 \).
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(b) The general solution then reads for arbitrary constants \( A, B \) and \( C \)

\[ u_n = (A + Bn + Cn^2)(-1)^n \ , \ n \geq 0 \]
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\[ u_n = (A + Bn + Cn^2)(-1)^n, \quad n \geq 0 \]

(c) To determine \( A, B \) and \( C \) through the use of the initial conditions, we set \( n \) in the solution expression in (b) to 0, 1 and 2 respectively, then

\[
\begin{align*}
  u_0 \text{ gives } & A = 1 \\
  u_1 \text{ gives } & (A + B + C)(-1) = 1 \\
  u_2 \text{ gives } & A + 2B + 4C = -7.
\end{align*}
\]

The solution of these 3 equations, \( A = 1, \ B = 0, \ C = -2 \), finally produces

the required particular solution

\[ u_n = (1 - 2n^2)(-1)^n, \quad n \geq 0 \]
General Procedure

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The general process for solving linear homogeneous recurrence relations with constant coefficients is like this:

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4. Use the initial conditions to find the specific solution.
Example

Solve the following (shifted Fibonacci) recurrence relation:

\[ f_n = f_{n-1} + f_{n-2}, \quad f_0 = 0, \text{ and } f_1 = 1 \]
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2. The roots are \( \lambda_1 = \frac{1 + \sqrt{5}}{2} \) and \( \lambda_2 = \frac{1 - \sqrt{5}}{2} \)
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4. Use the initial conditions to find the specific solution

\[ f_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n \]
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\[ a_n = 2(a_{n-1} - a_{n-2}) \]

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3. The general solution is \( a_n = A (1 + i)^n + B (1 - i)^n \)
4. Use the initial conditions to find the specific solution \( A = 1/2, B = 1/2, \)
   i.e., \( a_n = \frac{1}{2} (1 + i)^n + \frac{1}{2} (1 - i)^n \)
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\[ a_n = 6a_{n-1} - 9a_{n-2} \]

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3. The general solution is

$$a_n = A(1)^n + B(-1)^n + C\left(\frac{1}{2}\right)^n$$

4. Use the initial conditions to find the specific solution $A = 5/2$, $B = 1/6$, $C = -8/3$, so

$$a_n = \frac{5}{2} + \frac{1}{6}(-1)^n - \frac{8}{3}\left(\frac{1}{2}\right)^n$$
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  - begins with \( 0, 1 \), and ends with any \( n - 2 \) bit string.
More Examples

Example
How many binary strings (strings of ’0’s and ’1’s) of $n$ bits have no two consecutive zeros?

Solution.
- Let $a_n$ be the number of such strings.
- Clearly, $a_0 = 0$, $a_1 = 2$, and $a_2 = 3$.
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- So $a_n = a_{n-1} + a_{n-2}$.
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- The characteristic equation is \( \lambda^2 - \lambda - 1 \), etc.