

# Propositional Logic

## Argument Forms

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July 19, 2013



# Outline

# Order of Precedence

- 1 connectives within parentheses, innermost parentheses first
  - 2  $\sim$
  - 3  $\wedge, \vee$
  - 4  $\rightarrow$
  - 5  $\leftrightarrow$
- Some people actually place a higher precedence for  $\wedge$  than for  $\vee$ .
  - To avoid possible confusion we shall always insert the parentheses at the appropriate places.

# Basic concepts

- A logical argument is built from *propositions*, also called **statements**.
- A **proposition** is a sentence which is either true or false, e.g.,
  - ”*The first computer was built by Babbage.*”
  - ”*Dogs cannot see blue colour.*”
  - ”*Berlin is the capital of Albania.*”
- Propositions may be either
  - ▶ **asserted** (said to be true) or
  - ▶ **denied** (said to be false).
- The proposition is the meaning of the statement, not the particular arrangement of words used:
  - ▶ ”**beauty exists**” and ”**there exists beauty**” both express the same proposition.

# Basic concepts

- An **argument** is "a connected series of statements to establish a definite proposition" (Monty Python).
- There are three stages to an argument:
  - ① **premises** – assertions, often introduced by "because", "since", "obviously" and so on.
  - ② **inference** – deriving new propositions from premises
  - ③ **conclusion** – the final stage of inference, introduced by "therefore", "it follows that", "we conclude", etc.
- Typically an argument form will take the form

$$p_1, p_2, \dots, p_n, \therefore q$$

where propositions  $p_1, \dots, p_n$  are the premises and proposition  $q$  is the conclusion.

- The above argument form can also be represented vertically by

$$\begin{array}{ccc} p_1 & & p_1 \\ p_2 & & p_2 \\ \vdots & & \vdots \\ p_n & & p_n \\ \hline \therefore q & \text{or by} & \therefore q \end{array}$$

# Argument

## Definition. (Argument)

An **argument form** (argument) is a finite set of statements called **premises**, together with a single statement, called the **conclusion**, which the premises are taken to support.

- Here is an example of a (deductive) argument:
  - ▶ Every event has a cause. (**premise**)
  - ▶ The universe has a beginning. (**premise**)
  - ▶ All beginnings involve an event. (**premise**)
  - ▶ This implies that the beginning of the universe involved an event. (**inference**)
  - ▶ Therefore the universe has a cause. (**inference and conclusion**)
- The symbol  $\therefore$  is used to indicate an argument's conclusion. It is pronounced "*therefore*".
- Symbolic logic does not care about the contents of arguments.
- Symbolic logic studies the forms of arguments.

# Validity and invalidity

## Definition. (Validity and Invalidity)

An argument is **valid** if and only if its conclusion must be true when its premises are true. Otherwise, it is **invalid**.

The argument  $(p_1, p_2, \dots, p_n, \therefore q)$  is valid if and only if the proposition  $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$  is a tautology.

- The fact that an argument is valid does not imply that its conclusion holds.
  - ▶ This is because of the slightly counter-intuitive nature of implication.
- Obviously a valid argument can consist of true propositions.
- However, an argument may be entirely valid even if it contains only false propositions (no contradiction of the definition), e.g.,

*All insects have wings. (premise)*

*Woodlice are insects. (premise)*

*Therefore woodlice have wings. (conclusion)*

- The conclusion is not true because the premises are false. If the argument's premises were true, however, the conclusion would be true. The argument is thus entirely valid.

## Validity and invalidity (cont.)

- One can reach a true conclusion from one or more false premises, as in:

*All fish live in the sea. (premise)*

*Dolphins are fish. (premise)*

*Therefore dolphins live in the sea. (conclusion)*

- If the premises are false and the inference valid, the conclusion can be true or false.
- If the premises are true and the conclusion false, the inference must be invalid.
- If the premises are true and the inference valid, the conclusion must be true.

To say that an argument is valid is not to say that the conclusion is true, and

To say that an argument's conclusion is true is not to say that the argument is valid.

# Validity and invalidity (cont.)

- Notice that a valid argument may have false premises and a false conclusion.

## Practical Method

The validity of an argument can be tested through the use of the truth table by checking if **critical rows** (the ones where all premises are true) will correspond to the value **true** for the conclusion.

- An **invalid** argument is an argument which is not valid.
  - ▶ In other words, it has true premises but a false conclusion.

# Examples

## Example

1. Show that  $(p \vee q, p \rightarrow q, q \rightarrow r, \therefore r)$  is a valid argument.

- There are 3 variables (p, q, r) so we need  $2^3 = 8$  rows in our truth-table.

p	q	r
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

# Examples

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1. Show that  $(p \vee q, p \rightarrow q, q \rightarrow r, \therefore r)$  is a valid argument.

- There are 3 variables ( $p, q, r$ ) so we need  $2^3 = 8$  rows in our truth-table.

$p$	$q$	$r$	$p \vee q$	$p \rightarrow q$	$q \rightarrow r$
0	0	0	0	1	1
0	0	1	0	1	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	1	0	1
1	0	1	1	0	1
1	1	0	1	1	0
1	1	1	1	1	1

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p	q	r	$p \vee q$	$p \rightarrow q$	$q \rightarrow r$	
0	0	0	0	1	1	
0	0	1	0	1	1	
0	1	0	1	1	0	
0	1	1	1	1	1	← critical rows
1	0	0	1	0	1	
1	0	1	1	0	1	
1	1	0	1	1	0	
1	1	1	1	1	1	←

# Examples

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p	q	r	$p \vee q$	$p \rightarrow q$	$q \rightarrow r$
0	0	0	0	1	1
0	0	1	0	1	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	1	0	1
1	0	1	1	0	1
1	1	0	1	0	0
1	1	1	1	1	1

← critical rows correspond to 1 for r

- Valid arguments can have false conclusions.
- But a valid argument with a false conclusion must have a false premise.

# Examples

## Example

2. Show that  $(p \rightarrow q \therefore \neg p \rightarrow \neg q)$  is an invalid argument.

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$
0	0	1	1	1	1
0	1	1	0	1	0
1	0	0	1	0	1
1	1	0	0	1	1

# Examples

## Example

2. Show that  $(p \rightarrow q \therefore \neg p \rightarrow \neg q)$  is an invalid argument.

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$
0	0	1	1	<b>1</b>	<b>1</b>
0	1	1	0	<b>1</b>	<b>0</b>
1	0	0	1	0	1
1	1	0	0	<b>1</b>	<b>1</b>

← this critical row fails

# Examples

## Example

3. Show that  $(p \wedge q \therefore q)$  is a valid argument.

## Example

4. Show that  $(p \rightarrow q, p \therefore q)$  (Modus Ponens) is a valid argument.

- You can tell modus ponens is valid from the truth table for  $\rightarrow$ :

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- In only one row (the first) are both premises  $(p \rightarrow q)$  and  $p$  true. And, in this row,  $q$  is also true.

# Other Simple Argument Forms

Which of the following are valid?

❶  $(p \rightarrow q, q \therefore p)$

invalid

❷  $(p \rightarrow q, \sim q \therefore \sim p)$

valid

❸  $(p \rightarrow q, \sim p \therefore q)$

invalid

❹  $(p \rightarrow q, p \therefore \sim q)$

invalid

❺  $(p \rightarrow q, \therefore \sim p \rightarrow \sim q)$

invalid

# More Examples

- An argument form with contradictory premises is **valid**.

$$(p \wedge \sim p, \therefore q)$$

The premise formula is a contradiction! Always false. So there is no row on which the premise is true but the conclusion is false. Hence the argument form is valid, by our definition.

- An argument whose conclusion is a tautology is **valid**.

$$(p \rightarrow q, r \vee q, \therefore s \vee \sim s)$$

There is no row on which  $(s \vee \sim s)$  is false, so it is a tautology. Hence there is no row on which the premise formulas are true but the conclusion formula is false. Therefore the argument form is valid, by our definition.

# More Examples

- Argument A has five premises and one conclusion.  
All five premises are true, and the conclusion is true.  
Is argument A valid?
- *Answer.* Not enough information to tell. Whether an argument is valid does not depend on the fact that the premises and conclusion are true. Rather, it depends on whether its possible for the premises to be true and the conclusion false.
- Argument B has two premises and one conclusion.  
All two premises are true, and the conclusion is false.  
Is argument B valid?
- *Answer.* No. An argument is valid only if its impossible for the conclusion to be false when all the premises are true. Since you have been told that the premises are true and the conclusion is false, this is not impossible. So the argument is invalid.

# Converse, Inverse, and Contrapositive

- Given the implication  $p \rightarrow q$   
 $p$  is called the **premise** (or *hypothesis*) and  $q$  is called the **conclusion**.
- For any such a proposition  $p \rightarrow q$ , we also have
  - $q \rightarrow p$  the **converse** of  $p \rightarrow q$
  - $\sim p \rightarrow \sim q$  the **inverse** of  $p \rightarrow q$
  - $\sim q \rightarrow \sim p$  the **contrapositive** of  $p \rightarrow q$
- The inverse and the converse of an implication are not necessarily true.
- The contrapositive of an implication is always true.
- The following argument forms are invalid (aka fallacies):

$$p \rightarrow q \therefore q \rightarrow p \quad \text{invalid!}$$

$$p \rightarrow q \therefore \sim p \rightarrow \sim q \quad \text{invalid!}$$

- Fallacies** are mistakes of reasoning, as opposed to those that are of a factual nature.

# Examples

Write like

if p then q

- Maths teachers love their job.
  - ▶ let  $p = \text{"teaching maths"}$
  - ▶ let  $q = \text{"loving your job"}$
  - ▶ "If you teach maths, then you love your job"
  - ▶ its **converse**:  
"If you love your job, then you teach maths" (**false**)
  - ▶ its **inverse**:  
"If you don't teach maths, then you don't love your job" (**false**)
  - ▶ its **contrapositive**:  
"If you don't love your job, then you don't teach maths" (**true**)
- Try yourself: Students hate homework.

# To sum up

- **conditional:**

if  $p$  then  $q$

if **true** then

- **converse:**

if  $q$  then  $p$

**false**

- **inverse:**

if  $\sim p$  then  $\sim q$

**false**

- **contrapositive:**

if  $\sim q$  then  $\sim p$

**true**

# Rules of Inference

(Just another name for valid arguments)

modus ponens:

$$p \rightarrow q, p \therefore q$$

modus tollens:

$$p \rightarrow q, \sim q \therefore \sim p$$

disjunctive addition:

$$p \therefore p \vee q$$

conjunctive addition:

$$p, q \therefore p \wedge q$$

conjunctive simplification:

$$p \wedge q \therefore p$$

- Modus (ponendo) ponens [Latin] – the way that affirms by affirming
- Modus (tollendo) tollens [Latin] – the way that denies by denying

# Rules of Inference

(Just another name for valid arguments)

disjunctive syllogism:

$$p \vee q, \sim q \therefore p$$

hypothetical syllogism:

$$p \rightarrow q, q \rightarrow r \therefore p \rightarrow r$$

division into cases:

$$p \vee q, p \rightarrow r, q \rightarrow r \therefore r$$

rule of contradiction:

$$\sim p \rightarrow \perp \therefore p$$

contrapositive equivalence:

$$p \rightarrow q \therefore \sim q \rightarrow \sim p$$

- **Syllogism** – a deductive scheme of a formal argument consisting of a major (M) and a minor (m) premises, and a conclusion (C).

## Example

<i>All men are human;</i>	M
<i>All humans are mortal;</i>	m
<i>Therefore all men are mortal.</i>	C

$$p \vee q, q \rightarrow r, p \wedge s \rightarrow t, \sim r, \sim q \rightarrow u \wedge s, \therefore t$$

- 1 ▶  $q \rightarrow r$  premise
- ▶  $\sim r$  premise
- ▶  $\therefore \sim q$  by modus tollens
- 2 ▶  $p \vee q$  premise
- ▶  $\sim q$  by (1)
- ▶  $\therefore p$  by disjunctive syllogism
- 3 ▶  $\sim q \rightarrow u \wedge s$  premise
- ▶  $\sim q$  by (1)
- ▶  $\therefore u \wedge s$  by modus ponens
- 4 ▶  $u \wedge s$  by (3)
- ▶  $\therefore s$  by conjunctive simplification
- 5 ▶  $p$  by (2)
- ▶  $s$  by (4)
- ▶  $\therefore p \wedge s$  by conjunctive addition
- 6 ▶  $p \wedge s \rightarrow t$  premise
- ▶  $p \wedge s$  by (5)
- ▶  $\therefore t$  by modus ponens

$$p \rightarrow q \equiv \sim p \vee q$$

- We use a truth table to prove it

$p$	$q$	$\sim p$	$\sim p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

- We see that the last two columns (of interest!) are exactly the same, which means the two formulae are equivalent.

$$p \rightarrow q, q \rightarrow r, \therefore r$$

- We use a truth table to see if this argument is valid

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$r$
0	0	0	1	1	0
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	0	0
1	1	1	1	1	1

- The conclusion  $r$  fails at the first critical row, so the argument form is not valid.

$$r \rightarrow p, \sim p \vee q, s \rightarrow p \wedge r, \sim p \wedge \sim r \rightarrow s \vee t, \sim q, \therefore t$$

- We use the basic inference rules to prove that this argument is valid:

- 1  $\sim q, \sim p \vee q, \therefore \sim p$  (disjunctive syllogism)
- 2  $\sim p, r \rightarrow p, \therefore \sim r$  (modus tollens)
- 3  $\sim p, \sim r, \therefore \sim p \wedge \sim r$  (conjunctive addition)
- 4  $\sim p \wedge \sim r, \sim p \wedge \sim r \rightarrow s \vee t, \therefore s \vee t$  (modus ponens)
- 5  $\sim r, \therefore \sim r \vee p$  (disjunctive addition)
- 6  $s \rightarrow \sim p \wedge r, \therefore s \rightarrow \sim (\sim r \vee p)$  (De Morgan's law)
- 7  $\sim r \vee p, s \rightarrow \sim (\sim r \vee p), \therefore \sim s$  (modus tollens)
- 8  $s \vee t, \sim s, \therefore t$  (disjunctive syllogism)

hence the argument form is valid.

# A Problem

Can you decide who killed E?

One person in a gang comprised of four members (say, A, B, C, and D) killed a person named E. A detective obtained the following statements from the gang members (let's say  $S_x$  denotes the statement made by  $x$ , where  $x \in \{A, B, C, D\}$ ):

- 1  $S_A$ : B killed E.
- 2  $S_B$ : C was playing with A when E was knocked off.
- 3  $S_C$ : B didn't kill E.
- 4  $S_D$ : C didn't kill E.

The detective was then able to conclude that all but one were lying. Who killed E?

- Let us consider the following four statements denoted (1) to (4).
- The first two are true due to the detective's work:
  - (1) Only one of the statements  $S_A, S_B, S_C, S_D$  is true.
  - (2) One of A, B, C, D killed E.

## A Problem (cont.)

- From statements  $S_A$  and  $S_C$  we know that
  - (3)  $S_A \rightarrow S_D$  is true because if  $S_A$  is true, then B killed E which implies C did not kill E [due to (2)], implying  $S_D$  is also true.
- From (1), if  $S_A$  is true, then all other three sentences ( $S_A, S_C, S_D$ ) are false
  - (4)  $S_A \rightarrow \sim S_B \wedge \sim S_C \wedge \sim S_D$  is true.
- Let us examine the following sequence of statements:
  - (a)  $S_A \rightarrow S_D$  ( $S_A, \therefore S_D$ ) from (3)
  - (b)  $S_A \rightarrow \sim S_B \wedge \sim S_C \wedge \sim S_D$  from (4)
  - (c)  $\sim S_B \wedge \sim S_C \wedge \sim S_D \rightarrow \sim S_D$  conjunctive simplificant
  - (d)  $S_A \rightarrow \sim S_D$  hypothetical syllogism from (b), (c)
  - (e)  $\sim S_D \rightarrow \sim S_A$  modus tollens from (a)
  - (f)  $S_A \rightarrow \sim S_A$  hypothetical syllogism from (d), (e)
- Notice that (a) and (b) are premises, (c) - (e) were obtained from previous ones.
- We call a sequence of this type a **proof sequence**, with the last entry called a **theorem**.

## A Problem (cont.)

- The sequence (a) – (f) can be seen as special valid argument form, in which the truth of the first two statements will ensure the truth of the rest of them.
- We conclude that  $S_A \rightarrow \sim S_A$  is true, hence  $S_A$  must be false from the rule of contradiction: if  $S_A$  were true, then  $\sim S_A$  would be true, implying  $S_A$  is false: contradiction.
- From the definition of  $S_C$  we see that  $\sim S_A \rightarrow S_C$
- From modus ponens  $\sim S_A \rightarrow S_C, \sim S_A, \therefore S_C$  we conclude  $S_C$  is true.
- Since (1) gives  $S_C \rightarrow \sim S_A \wedge \sim S_B \wedge \sim S_D$  we obtain by using conjunctive simplification  
 $(S_C \rightarrow \sim S_A \wedge \sim S_B \wedge \sim S_D, \sim S_A \wedge \sim S_B \wedge \sim S_D \rightarrow \sim S_D, \therefore S_C \rightarrow \sim S_D)$   
that is  $S_C \rightarrow S_D$
- Finally, from modus ponens  $(S_C \rightarrow \sim S_D, S_C, \therefore \sim S_D)$  we conclude  $\sim S_D$  is true, that is **C killed E**.
- Verbal arguments in this case are more concise. Can you find them?