Algorithms

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Outline

1. Introduction
2. Examples
3. Algorithm Analysis
4. Searching Algorithms
5. Sequential Searching
   - The floor/ceiling functions
6. Binary Search
7. Complexity of Binary Search
8. Performance of Sorting Algorithms

**algorithm**, [alго-ridhм], n. a rule for solving a mathematical problem in a finite number of steps.
[Root: Late Latin *algorismus*, from the Arabic name Al-Khowarizmi, a 9th century mathematician Abu Jafar Mohammed ben Musa.]

A computer program is simply an implementation of an algorithm on a computer.
Examples

- How to behave on a date

```java
public void smartSwap(int ref a, int ref b) {
    a += b;
    b = a - b;
    a -= b;
}
```
Examples

• How to behave on a date
  
  WHILE wondering how to get out without being rude
     DO tell him he’s a wonderful guy
  ENDWHILE

• How to get out of bed
  if mother is watching
      then scan her for weapons, especially those prohibited by Geneva Convention
      if result is affirmative
          then begin negotiations

• How to swap two numbers
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Examples

Euclid’s Algorithm

while m is greater than zero:
   If n is greater than m, swap m and n.
   Subtract n from m.

n is the GCD

Example

Program in C

```c
int gcd(int m, int n)
/* precondition: m>0 and n>0. Let g=gcd(m,n). */
{
    while( m > 0 )
    { /* invariant: gcd(m,n)=g */
        if( n > m )
        { int t = m; m = n; n = t; } /* swap */
        /* m >= n > 0 */
        m -= n;
    }
    return n;
}
```
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Program in Prolog

; The goal gcd(I, J, K) succeeds when the greatest common
; divisor of I and J is K.
gcd(I,0,I).
gcd(I,J,K) :- R is I mod J, gcd(J,R,K).

Example

Program in Java

```java
public static long gcd(long a, long b){
    long factor= Math.max(a, b);
    for(long loop= factor;loop > 1;loop--){
        if(a % loop == 0 && b % loop == 0){
            return loop;
        }
    }
    return 1;
}
```
Example

Program in Prolog

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**Motivation**

- Efficient algorithms lead to efficient programs.
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- Efficient programs make better use of hardware.
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- Efficient programs sell better.
- Efficient programs make better use of hardware.
- Programmers who write efficient programs are more marketable than those who don’t!
Computational complexity theory is the study of the cost of solving interesting problems.
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- Measure the amount of resources needed
  - Upper bounds: give a fast algorithm
  - Lower bounds: no algorithm is faster

Algorithm analysis is the analysis of resource usage of given algorithms.

- Exponential resource use is bad.
  - Make resource usage a polynomial
  - Make that polynomial as small as possible
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Algorithm analysis

1. the input and the output of the algorithm should be described

2. The input is the set \( \{1, 2, 3, \ldots, n\} \) and the output is the sum of these \( n \) integers.

3. The algorithm has one rule, the formula

\[
1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}
\]

We gave a proof by induction to this rule.
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Example: describe and analyse an algorithm which computes the sum of first $n$ positive integers

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Searching Algorithms

- General definition
  - Locate an element $x$ in a list of distinct elements $a_1, a_2, \ldots, a_n$, or
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    * determine that it is not in the list.
  - Return the position $k$ at which $a_k$ matches $x$ or, otherwise
    * return a phantom position “$-1$” (or “$o$“) used to denote that $x$ is not found.
Sequential (Linear) Searching

Linear Searching

**Input:** unsorted sequence \( a_1, a_2, \ldots, a_n \)
- position of target value \( x \)

**Output:** subscript of entry equal to target value; 0 if not found

*Initialize:* \( i \leftarrow 1 \)

*while* \( (i \leq n \text{ and } x \neq a_i) \)

- \( i \leftarrow i + 1 \)

*if* \( i \leq n \) *then* \( \text{location} \leftarrow i \)
*else* \( \text{location} \leftarrow 0 \)

Let \( S(n) \) be the number of comparisons needed in the worst case to complete the linear search.
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\text{else location } \leftarrow 0
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- Let $S(n)$ be the number of comparisons needed in the worst case to complete the linear search.
- The worst case is when the searched item is not found or it is in the last position, that is $S(n) = n$
The floor/ceiling of a real number

Given $x \in \mathbb{R}$,

- the floor of $x$, denoted by $\lfloor x \rfloor$, is the greatest integer not exceeding $x$.

- the ceiling of $x$, denoted by $\lceil x \rceil$, is the smallest integer not less than $x$. 

Examples

If $x \in \mathbb{Z}$, then $\lfloor x \rfloor = \lceil x \rceil = x$.

$\lfloor 3.2 \rfloor = 3$, $\lfloor -4.5 \rfloor = -5$, and $\lfloor -4 \rfloor = -4$.

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Ioan Despi – Discrete Mathematics 12 of 20
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Binary Search

- Binary search is a divide-and-conquer (divide et impera) strategy.

1. Define two counters $i$ and $j$ and set $i = 1$ and $j = n$ as initial values.
2. If $j < i$, the algorithm terminates without finding $a$.
3. If not, set $k = \lfloor (i + j) / 2 \rfloor$.
4. If $a < a_k$, go to (4). If $a > a_k$, go to (5). If $a = a_k$, the algorithm found $a$.
5. Set $i = k + 1$ and repeat (2).
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- Which sublist is to be continued for the search depends on whether the given KEY \( (a) \) is after or before the item at the middlemost position.
- Continue the binary search on the correct sublist until either a match is found, or the latest sublist is reduced to an empty list.
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4. Set \( j = k - 1 \) and repeat (2)
5. Set \( i = k + 1 \) and repeat (2)
Binary Search

- If a list \( a_m, a_{m+1}, \ldots, a_n \) is indexed from \( m \) to \( n \), then the middlemost position is at \( k = \left\lfloor \frac{m+n}{2} \right\rfloor \).
Binary Search

- If a list $a_m, a_{m+1}, \cdots, a_n$ is indexed from $m$ to $n$, then the middlemost position is at $k = \left\lfloor \frac{m+n}{2} \right\rfloor$.
- Suppose a given list $I(1), \cdots, I(n)$ is sorted in the increasing order, then the binary search algorithm can be rephrased as follows.

1. $F = 1, L = n$. /* $F$ = first index, $L$ = last index */
2. while $F \leq L$ do
   (i) find middlemost position $k = \left\lfloor \frac{F+L}{2} \right\rfloor$
   (ii) if KEY = $I(k)$ then /* match found */ output $k$ and stop
      else if KEY > $I(k)$ then /* move to 2nd half of the list */
      $F = k + 1$ /* redefining starting position */
      else /* move to 1st half of the list */
      $L = k - 1$ /* redefining end position */
3. output the phantom position 
   
Ioan Despi – Discrete Mathematics 14 of 20
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**Binary Search**

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2. while $F \leq L$ do
   (i) find middlemost position $k = \left\lfloor \frac{F + L}{2} \right\rfloor$
   (ii) if $\text{KEY} = I(k)$ then /* match found */
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        else if $\text{KEY} > I(k)$ then /* move to 2nd half of the list */
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3. output the phantom position “−1”.

Ioan Despi – Discrete Mathematics 14 of 20
Example

Find 13 from the ordered list \{1, 3, 5, 7, 9, 13, 15, 17, 19\} with both the binary search and the sequential search. Give the number of comparisons needed in both cases.

**Binary search.** 3 comparisons are needed as can be seen from the diagram below:

```
\begin{array}{cccccccc}
1 & 3 & 5 & 7 & 9 & 13 & 15 & 17 & 19 \\
I(1) & I(3) & I(5) & I(7) & I(9) & 13 & 15 & 17 & 19 \\
\end{array}
```

```plaintext
middlemost: 5 = \left\lfloor \frac{1+9}{2} \right\rfloor \\
9 < 13: i = 6, 7, 8, 9 \\
13 = 13: match found in 3 comparisons
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\[
\begin{array}{cccccccc}
  i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
  I(i) & 1 & 3 & 5 & 7 & \boxed{9} & 13 & 15 & 17 & 19 \\
\end{array}
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middlemost: \[ 5 = \left\lfloor \frac{1 + 9}{2} \right\rfloor \quad 9 < 13 : \]

<table>
<thead>
<tr>
<th>(i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I(i))</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>19</td>
</tr>
</tbody>
</table>

middlemost: \[ 7 = \left\lfloor \frac{6 + 9}{2} \right\rfloor \quad 15 > 13 : \]
Example

Find 13 from the ordered list \( \{1, 3, 5, 7, 9, 13, 15, 17, 19\} \) with both the binary search and the sequential search. Give the number of comparisons needed in both cases.

**Binary search.** 3 comparisons are needed as can be seen from the diagram below.

middlemost: \( 5 = \left\lfloor \frac{1+9}{2} \right\rfloor \) \( 9 < 13 : \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>19</td>
</tr>
</tbody>
</table>

middlemost: \( 7 = \left\lfloor \frac{6+9}{2} \right\rfloor \) \( 15 > 13 : \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I(i) )</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>19</td>
</tr>
</tbody>
</table>
## Example

Find 13 from the ordered list \{1, 3, 5, 7, 9, 13, 15, 17, 19\} with both the binary search and the sequential search. Give the number of comparisons needed in both cases.

### Binary search.

3 comparisons are needed as can be seen from the diagram below.

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I(i)$</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>19</td>
</tr>
</tbody>
</table>

**middlemost:** $5 = \left\lfloor \frac{1 + 9}{2} \right\rfloor$  \hspace{1cm} $9 < 13$:

**middlemost:** $7 = \left\lfloor \frac{6 + 9}{2} \right\rfloor$  \hspace{1cm} $15 > 13$:

$13 = 13$: match found in 3 comparisons

<table>
<thead>
<tr>
<th>i</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I(i)$</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>19</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>i</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I(i)$</td>
<td>13</td>
</tr>
</tbody>
</table>
Example

Find 13 from the ordered list \{1, 3, 5, 7, 9, 13, 15, 17, 19\} with both the binary search and the sequential search. Give the number of comparisons needed in both cases.

Sequential search. 6 comparisons are needed in this case, see below:

\[
\begin{array}{cccccccc}
  i & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
  I(i) & 1 & 3 & 5 & 7 & 9 & 13 & 15 \\
\end{array}
\]

-→ search stops after finding the match after 6 comparisons
Example

Find 13 from the ordered list \{1, 3, 5, 7, 9, 13, 15, 17, 19\} with both the binary search and the sequential search. Give the number of comparisons needed in both cases.

**Sequential search.** 6 comparisons are needed in this case, see below:

<table>
<thead>
<tr>
<th>(i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I(i))</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>19</td>
</tr>
</tbody>
</table>

\[\rightarrow\]

search stops after finding the match after 6 comparisons
Complexity of Binary Search

Let $B(n)$ be the maximum number of comparisons the binary search needs to complete the search for an ordered list of $n$ items.
Then, for $m, n \in \mathbb{N}$ with $m \geq 1$,

(i) $B(1) = 1$
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Let $B(n)$ be the maximum number of comparisons the binary search needs to complete the search for an ordered list of $n$ items.
Then, for $m, n \in \mathbb{N}$ with $m \geq 1$,

(i) $B(1) = 1$

(ii) $B(m) \leq B(n)$ if $m \leq n$
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(i) $B(1) = 1$

(ii) $B(m) \leq B(n)$ if $m \leq n$

(iii) $B(2m) = 1 + B(m)$

see: $ \underbrace{\ast \ldots \ast}_{m-1} \ast \underbrace{\ast \ldots \ast}_{m}$

Hence we have $B(2^k) = B(2^{k-1}) + 1 = B(2^{k-2}) + 2 = \ldots = B(2^0) + k = 1 + k$ for $k \geq 0$.

For any $n \in \mathbb{N}$ with $n \geq 1$, there exists $k \in \mathbb{N}$ such that $2^k \leq n < 2^{k+1}$.

Taking a log $\log_2$ on both sides of the inequality we obtain $k \leq \log_2 n < k + 1$.

Hence $k = \lfloor \log_2 n \rfloor \iff 2^k \leq n < 2^{k+1}$. 
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Let $B(n)$ be the maximum number of comparisons the binary search needs to complete the search for an ordered list of $n$ items.

Then, for $m, n \in \mathbb{N}$ with $m \geq 1$,

(i) $B(1) = 1$

(ii) $B(m) \leq B(n)$ if $m \leq n$

(iii) $B(2m) = 1 + B(m)$ see:

(iv) $B(2m + 1) = 1 + B(m)$

Hence we have

$B(2^k) = B(2^k - 1) + 1 = B(2^k - 2) + 2 = \ldots = B(2^0) + k = 1 + k$ for $k \geq 0$. 

For any $n \in \mathbb{N}$ with $n \geq 1$, $\exists k \in \mathbb{N}$ such that $2^k \leq n < 2^{k+1}$.

Taking a log base 2 on both sides of the inequality we obtain $k \leq \log_2 n < k + 1$.

Hence $k = \lfloor \log_2 n \rfloor$ ⇔ $2^k \leq n < 2^{k+1}$. 

\[ \begin{array}{c}
\{ \{ \} \} \quad \{ \} \\
\{ \} \end{array} \]

\[ \begin{array}{c}
\{ \} \quad \{ \} \\
\{ \} \end{array} \]
Complexity of Binary Search

Let $B(n)$ be the maximum number of comparisons the binary search needs to complete the search for an ordered list of $n$ items.

Then, for $m, n \in \mathbb{N}$ with $m \geq 1$,

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(ii) $B(m) \leq B(n)$ if $m \leq n$

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Hence we have

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For any $n \in \mathbb{N}$ with $n \geq 1$, $\exists k \in \mathbb{N}$ such that $2^k \leq n < 2^{k+1}$.

Taking a $\log_2$ on both sides of the inequality we obtain $k \leq \log_2 n < k + 1$.

Hence

$$k = \lfloor \log_2 n \rfloor \iff 2^k \leq n < 2^{k+1}.$$
Complexity of Binary Search

From (ii) we thus obtain

\[ B(2^k) \leq B(n) \leq B(2^{k+1}) \]

which gives \( k + 1 \leq B(n) \leq k + 2 \), or simply

\[ \lfloor \log_2 n \rfloor + 1 \leq B(n) \leq \lfloor \log_2 n \rfloor + 2. \]

Hence for \( n \geq 4 \), we have \( \log_2 n \geq 2 \) and \( \log_2 n \leq \lfloor \log_2 n \rfloor + 1 \) for \( n \geq 1 \) so

\[ \log_2 n \leq B(n) \leq 2 \log_2 n. \]

Thus \( B(n) = \mathcal{O}(\log_2 n) \) and \( \log_2 n = \mathcal{O}(B(n)) \).

We note that in the literature of computer science, \( \log_2 n \) is often abbreviated to \( \log n \).
Complexity of Binary Search

Our analysis shows that binary search can be done in time proportional to the log of the number of items in the list.

<table>
<thead>
<tr>
<th>n</th>
<th>( \log_2 n )</th>
<th>( n )</th>
<th>( 2^n )</th>
<th>( n! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
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<tr>
<td>2</td>
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</tr>
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<td>8</td>
<td>200</td>
<td>1.6E+30</td>
<td>too big</td>
</tr>
</tbody>
</table>

In Python:

```
>>> 2**500
3273390607896141870013189696827599152216642043064789483291368096133796404674554883270092325904157150886684127560071009217256545885393053328527589376L
```
Complexity of Binary Search

- Our analysis shows that binary search can be done in time proportional to the log of the number of items in the list.
- This is considered very fast when compared to linear or polynomial algorithms.
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<table>
<thead>
<tr>
<th>n</th>
<th>log (n)</th>
<th>(n^2)</th>
<th>(n^1)</th>
<th>(n^2)</th>
<th>(n^3)</th>
<th>(n^4)</th>
<th>(n^5)</th>
<th>(n^6)</th>
<th>(n^7)</th>
<th>(n^8)</th>
<th>(n^9)</th>
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</table>

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<tbody>
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<tr>
<td>5000</td>
<td>13</td>
<td>25E+06</td>
<td>too big</td>
</tr>
</tbody>
</table>

In python:

```python
amth140@turing> python
Python 2.7.3 ...
>>> 2**500
3273390607896141870013189696827599152216642046
0430647894832913680961337964046745548832700923
25904157150886684127560
071009217256545885393053328527589376L
```
Performance of Sorting Algorithms

- A number of sorting algorithms will be introduced in the tutorials over the semester:
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- All sorting algorithms require a total of $O(n^2)$ comparisons in the corresponding worst (i.e., the most “laborious”) cases.
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- All sorting algorithms require a total of $\mathcal{O}(n^2)$ comparisons in the corresponding worst (i.e., the most “laborious”) cases.
- Let $M(n)$ be number of comparisons needed for the merge sort in the worst case, and $\overline{Q}(n)$ be the average number of comparisons needed for the quick sort, then both $M(n)$ and $\overline{Q}(n)$ are $\mathcal{O}(n \log_2 n)$. 
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• Let \( M(n) \) be number of comparisons needed for the merge sort in the worst case, and \( \overline{Q}(n) \) be the average number of comparisons needed for the quick sort, then both \( M(n) \) and \( \overline{Q}(n) \) are \( \mathcal{O}(n \log_2 n) \).

• More precisely, however, one can show for sorting a list of \( n \) (\( \geq 1 \)) items

\[
M(n) \leq 4n \log_2 n, \quad \overline{Q}(n) \leq (2 \ln 2)n \log_2
\]