University of New England
Faculty of Arts and Sciences
School of Science & Technology

AMTH140 – Discrete Mathematics

Unit Information

Semester 2, 2012

Ioan Despi

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Welcome to Discrete Mathematics

Welcome to amth140!
Here you are some vital statistics:
Our school has its own server called turing.
Moodle server’s entry for AMTH140 contains just a link to turing, that is:
all the information regarding this unit can be found on
http://turing.une.edu.au/~amth140

Administrative Details

Unit Title: Discrete Mathematics

Unit Code: AMTH140

Semester: 2, 2012

Contacts:

Unit code: amth140
e-mail: amth140@turing.une.edu.au

Lecturer: Dr. Ioan Despi
office: 107 Booth Block
phone: (02) 6773 2513
fax: (02) 6773 3312

Tutor: Ms. GuYin Song
School: (02) 6773 5022
e-mail: admin-st@une.edu.au
fax: (02) 6773 5011

Lectures
Monday 13:00 – 14:00 B251
Thursday 12:00 – 13:00 B264
Friday 10:00 –11:00 B264

Tutorials
Wednesday 2:00 – 3:00 p.m. B264
Wednesday 3:00 – 4:00 p.m. B264
Wednesday 4:00 – 5:00 p.m. B264

Practical
Tuesday 11:00 – 12:00 a.m. B264
Thursday 2:00 –3:00 p.m. B264

Consultation times
10:00 - 12:00 a.m. Monday
11:00 - 12:00 a.m. Friday

All students must attend 3 Lectures, 1 Practical/Laboratory Class, and 1 Tutorial each week.
Practical and Tutorial classes start in the second week.
Aims and Prerequisites

Discrete Mathematics introduces you to a wide range of terminology and tools that have particular use in computer science. Topics covered in this unit include Sets, Mathematical Induction, Big O Notation, Logic, Predicate Calculus, Graphs, Linear Recurrence Relations, Sorting Method, Relations and Partial Ordering.

As a guide, you should allocate ten hours per week private study time for this unit. There will be lecture notes appearing on the web & the Internet as the unit proceeds. So you should pay close attention to the amth140 web pages on turing.

Learning Outcomes (LO) and Graduate Attributes (GA)

Upon completion of this unit, students will be able to develop ideas that provide a way of analysing the physical world; undertake problem identification, formulation and solution; become proficient in using new tools that are useful in Computer Science, that is

LO 1 demonstrate competency and fluency in the terminology and notation of discrete mathematics

LO 2 solve problems based on the application of concepts relating to discrete or finite mathematics

LO 3 obtain the ability to think scientifically.

UNE graduate attributes

UNE has a policy that identifies the special attributes of a UNE graduate. The policy can be found at http://www.une.edu.au/gamanual/ It is expected that, during the course of your undergraduate degree, you will develop these attributes in conjunction with your discipline knowledge. Those addressed by this unit are reflected in the unit learning outcomes and assessment tasks. You can assess your developing skill level after each unit by using the self-reflection guide and resources located at

http://www.une.edu.au/gamanual/students

This unit addresses the following graduate attributes (GA):

GA 1 Knowledge of discipline: Students will be taught the knowledge useful in computer sciences in this unit and the students are expected to use them to solve practical problems.

GA 2 Communication Skills: Students will be taught on how to build up their ability to to use mathematics through their life long learning.
GA 6 Problem Solving: Students will be taught to use their knowledge in mathematics to solve problems arising from computer science.

Intensive school

There is no intensive school for this unit.

Textbooks

Lecture notes will be provided. The lecture notes cover all the material required of this unit. While the booklet of lecture notes is more concise, the below mentioned textbooks provide you with extra reading and insights. Therefore the lecture notes and the textbooks can be used to complement each other.


Recommended Readings


From time to time the lecturer will post on the AMTH140 web pages the slides he used to teach a particular lecture, more exercises and examples, etc. Check UnitLinks [1] if hungry for more information.

How to study the unit

It is essential that you read carefully the lecture notes and try to understand everything presented there. On the other hand, assignments are an essential part of this unit. It is important to notice that through doing the assignments and through the feedbacks from your marked assignments, you can check your understanding of the materials in this unit in a timely fashion. Do not get too far behind and do not leave important materials unfinished before going to the next step.

Assignments & Assessment

You must complete all assessment tasks to pass the unit. There will be 6 assignments for students enrolled in amth140. Each assignment is worth 100 marks. The exam is worth 100 marks. The assignments represent 30% from the final mark (so each assignment will bring at most 5% to the final mark), and the exam represents the other 70%.

To pass the unit you must get at least 50% in each component, that is

- 300 marks (representing 15% in the final mark) in the assignments part and
- 50 marks (representing 35% in the final mark) in the exam.

See the separate Assessment booklet for information about the assessment for this unit.

At the School Teaching and Learning Committee meeting on 9 February 2012 the Late Assessment Policy for this School was amended and now reads:

School Policy on Late Assignments

For assessment tasks worth at least 10% of the overall mark for the unit, a penalty of 5% per working day will apply. This penalty rate applies for a maximum of ten days, after which no mark will be awarded. A period of grace of 7 days from the due date will apply to externals for assignments posted to the University from remote locations. An extension for an Online Quiz may be granted by the Unit Coordinator but cannot be extended beyond the date by which solutions have been posted. Consideration will be given for extenuating personal circumstances, but evidence must be supplied.

Grounds for extensions may be considered based on

1. Medical certificate or obvious illness; or

2. Extenuating personal circumstances, which must be accompanied by documentary evidence, eg, certificate from counsellor.

Notes

- No submission will be accepted after the end of the trimester, that is after Friday, 28th of September, 2012.

- Asking unit coordinators for preliminary review of any assessment tasks prior to formal submission is inappropriate and unfair to other students without that opportunity.

- Students may request that an assessment task be re-marked, in its original form, in circumstances where the student presents a case arguing that the original marking was unfair or inconsistent with marking guidelines. This request must be directly addressed to the
unit coordinator, with a copy to the Head of School, by the student within 10 working
days of receipt of the original marked assessment task.

- Information regarding all aspects of assessment can be found at

- Information about special assessment (Special Examinations, Special Extension of Time)
  can be found at

- In any situation you have to submit the assignments prior to the end of trimester.

- To request an extension, you will need to email amth140@mcs.une.edu.au, prior to the
due date, and provide the following:
  
  – your name,
  
  – the assignment for which you seek an extension,
  
  – the grounds for granting an extension, and
  
  – the date you anticipate being able to submit your assignment.

**UNEonline**

UNEonline is UNEs online teaching system. As a current student you have access to the stu-
dent portal [myUNE](http://www.une.edu.au/policies/pdf/assessment.pdf), through which you can access your online units, manage your enrolment,
change personal information and access many useful functions. Log in to myUNE from the
UNE homepage using your UNE username and password. All units have an online site that
may include features such as a message board, a discussion forum and electronic downloads of
教学 material.

**Accessing the online site for your other units**

To access the online sites for most UNE units (stored on a Moodle server), log in to myUNE from
the UNE homepage using your UNE username and password. Go to the myStudy tab. A list
of the units in which you are currently enrolled will be displayed in the **myUnits** and **Services**
table. When the online site for one of your units becomes available, a UNEonline column will
be displayed. You can access the online unit by clicking on the icon in this column. Please
note that most online units will not be available until the first day of teaching. If you have any
problems related to accessing myUNE or the online site for your unit, contact the IT Service Desk
on 02 6773 5000, via AskUNE or by emailing servicedesk@une.edu.au.
List of usernames disclaimer

Please note that your username within UNEonline appears as part of a list in some places. Only students enrolled in the unit have access to this list. Please contact your unit coordinator if you have any concerns.

General Information

Student Centre

The Student Centre provides you with a focal point of contact for all your administrative enquiries during your study at UNE, including selecting units and managing your enrolment. If you have administrative enquiries relating to your study at UNE, go to the Student Centre page at StudentCentre. Alternatively, you can go to AskUNE and submit a question by clicking on the Contact Us tab. You can also find information on all aspects of studying at UNE on the Current Students page at Current Students (UNE Contacts Page updated: http://www.une.edu.au/contacts/#is)

Library Services

The UNE University Library has an extensive collection of books, journals and online resources.

What can the Library do for you?

Find out at http://www.une.edu.au/library/services/unit_guide.php. This guide provides easy pathways to UNEs vast online resources and shows how the Library can help you with your studies. You can borrow books, obtain copies of articles and exam papers, and request advice from librarians on search strategies and information tools to use.

Learning support

The Academic Skills Office (ASO) is UNE’s learning support unit. The ASO has study skills advisors and a wealth of print and online resources to help you with your study skills development or problems.

http://www.une.edu.au/studentcentre
http://www.une.edu.au/askune
http://www.une.edu.au/for/current-students/
ASO fact sheets

The ASO has developed a series of fact sheets that answer the questions most frequently asked by students. They can be found at

http://www.une.edu.au/tlc/aso/students/factsheets/

ASO discussion forum

If you would like to discuss specific issues related to study skills or academic writing with an advisor, or benefit from the questions other students ask, you can log on to the ASO Discussion Forum at


Key terms

A glossary of some of the key terms used to describe academic and administrative activities, roles and structures at UNE can be found at

http://www.une.edu.au/policies/pdf/glossarykeyterms.pdf  It is very useful for understanding the terminology associated with your study.

Other Support at UNE

Other support services are available to assist you throughout the course of your studies. Some of these services are outlined below. For the full range of support services, go to http://www.une.edu.au/for/current-students/ and follow the links.

Student Assist

Student Assists support services include disability and special needs support, counselling, and career development. To see the range of services they offer, go to http://www.une.edu.au/student-assist/

Aboriginal and Torres Strait Islander students

The Oorala Aboriginal Centre is a study support and advisory centre for internal and external Aboriginal and Torres Strait Islander students at UNE. To find out more about the support services Oorala offers, go to

http://www.une.edu.au/ororal/
International students

International Services provides support for international students and provides a link between the administrative and academic functions at UNE. See http://www.une.edu.au/elis for more information about the services offered, and for an e-copy of the International Student Handbook go to http://www.une.edu.au/elis/brochures/

AskUNE

If you have questions that are not answered by this booklet, go to AskUNE, http://www.une.edu.au/askune/ At AskUNE you can find answers to many common enquiries or submit an enquiry of your own by clicking on the ‘Contact Us’ tab.

PLAGIARISM

Students are warned to read the statement in the Faculty’s Undergraduate and Postgraduate Handbooks for 2011 regarding the University’s Policy on Plagiarism. Full details of the Policy on Plagiarism are available in the UNE Handbook and at the following site:


Please read carefully


and


In addition, you must complete the Plagiarism Declaration Form for all assignments, practical reports, etc. submitted in this unit.

For electronic submission of assignments, it is presumed that you have read the web site and have agreed with the conditions so you don’t have to submit the form.

Plagiarism is the action or practice of taking and using as one’s own the thoughts or writings of another without acknowledgment. The following practices constitute acts of plagiarism and are a major infringement of UNE’s academic values:

- where paragraphs, sentences, a single sentence or significant parts of a sentence are copied directly, are not enclosed in quotation marks and appropriately referenced;

- where direct quotations are not used, but are paraphrased or summarised, and the source of the material is not referenced within the text of the paper; and
• where an idea which appears elsewhere in any form is used or developed without reference being made to the author or the source of that idea.

It is your responsibility to:

• read, understand and comply with the policy on Plagiarism and Academic Misconduct found at the website above;

• familiarise yourself with the conventions of referencing for your discipline(s);

• avoid all acts which could be considered plagiarism or academic misconduct;

• seek assistance from appropriate sources if you become aware that you need more knowledge and skills in relation to academic writing;

• be aware that when you submit an assignment through the University's e-Submission system, you are deemed to have signed the plagiarism declaration form;

• submit a separate signed and dated plagiarism declaration form with every task, report, dissertation or thesis submitted in hard copy for assessment or examination.

You should refer to the following websites for further advice and assistance:

• Avoiding Plagiarism and Academic Misconduct (Coursework): Information for Students
  This information explains the principles of good scholarship and has guidelines to help you avoid plagiarism. It also has guidelines for referencing and research, and advice on the use of internet sites.

• Academic Skills Office,
  The Academic Skills Office has a variety of support materials to assist you with referencing and avoiding plagiarism.

• eSKILLS UNE Keeping Track,
  eSkills Keeping Track has advice about organising your information for assignments and on referencing appropriately.

Some examples of this are books, journals, WWW material, theses, computer stored data and software, lecture notes or tapes.
**TurnItIn**

UNE uses a software application to determine the originality of assessable work submitted by its students. This software is called TurnItIn and it is part of the submission process.

When a file is submitted to TurnItIn, the software compares the text in the submitted files with text from a range of electronic sources including online journals, online databases, the Internet and the TurnItIn database. Any strings of text that occur in both the submitted document and in one or more of the electronic sources are identified by the software with a unique number and colour in what TurnItIn calls the ‘originality report’.

More information about e-Submission and TurnItIn can be found at [http://www.une.edu.au/tlc/students/services/esub-tii.php](http://www.une.edu.au/tlc/students/services/esub-tii.php)

**Examinations**

The Examinations page at [http://www.une.edu.au/exams/](http://www.une.edu.au/exams/) has important information about examinations, including your responsibilities as a student in relation to exams, information about examination dates and special exams, and links to who to contact if you have queries.

**Complaints and Appeals**


University of New England
Faculty of Arts and Sciences
School of Science & Technology

AMTH140 – Discrete Mathematics

Tutorials and Practice Classes
QUESTIONS

Semester 2, 2012

Ioan Despi
TUTORIAL 1
The Bubble Sort

Question 1

One of the most important and frequently used tasks in computer science is that of sorting a file. There are many methods of sorting. Perhaps the simplest – but not necessarily the best – is the bubble sort. It makes little difference what kind of items are in the list. For example they could be numbers or names. We shall consider a list of names. Our job is to put the list in alphabetical order.

The basic idea in the bubble sort is to pass through the list comparing each item with the next and interchanging them if they are out of order. If at the end of the pass the list is in the correct order, nothing further needs to be done. If not, another pass is made through the list and adjacent names are interchanged as necessary. The passes are repeated until no name is out of order. The process is indicated below. Observe how particular names bubble upward and others sink downward to their correct position. Indeed, after the first pass the last name in the list must be in its correct position. So if a second pass is necessary, we do not bother to compare the last item in the list with the one above it. After the second pass the last two items in the list must be in their correct position, and so on. Thus the process must stop after at most \( n - 1 \) passes.

Example

<table>
<thead>
<tr>
<th>Original Lists</th>
<th>List after Pass 1</th>
<th>List after Pass 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6 exchanges done</td>
<td>5 exchanges done</td>
</tr>
</tbody>
</table>

| Kris           | Kris            | Anita           |
| Nicole         | Anita           | Amelia          |
| Anita          | Amelia          | Kris            |
| Amelia         | Nicole          | Deborah         |
| Sandra         | Deborah         | Nicole          |
| Deborah        | Paula           | Maria           |
| Paula          | Maria           | Louise          |
| Maria          | Louise          | Paula           |
| Louise         | Sandra          | Sandra          |

(i) Complete the above process until the list is in alphabetical order.

Questions marked by stars “*” in this booklet are optional and are usually difficult. Avoid such questions completely if you happen to experience some difficulties with them.
The bubble sort algorithm is a particularly simple sorting algorithm, but there are other algorithms which in general involve fewer steps and which are therefore quicker to execute. The bubble sort has its worst case when the list happens to be in reverse alphabetical order. We shall now examine this worst case.

Let \( L_n \) be a list of \( n \) names in reverse alphabetical order. In the sorting process, each comparison of adjacent names is called a step. Let \( B(n) \) be the number of steps to put this list in correct order.

(ii) Let \( L_n \) be \( a_1, a_2, \ldots, a_n \). Write down the list after one pass.

(iii) How many comparisons have been made in this first pass?

Observe that \( a_1 \) is now in its correct position, and \( a_2, a_3, \ldots, a_n \) is in reverse alphabetical order. So this list of \( n - 1 \) names requires \( B(n - 1) \) steps to put it in alphabetical order.

(iv) So \( B(n) = B(n - 1) + \ldots \).

(v) Observing that \( B(1) = 0 \), use iteration to find \( B(10) \).

(vi) Using induction, prove that \( B(n) = \frac{1}{2}(n^2 - n) \).

(vii) Use (vi) to verify your answer in (v).
TUTORIAL 2
Divide and Conquer

It is sometimes possible to divide a problem into smaller subproblems, solve the subproblems, and then combine their solutions to obtain the solution to the original problem. This general approach, which is a very common one in computer science, is called divide and conquer. Since the subproblems are usually of the same type as the original problem, a divide and conquer strategy can often be implemented as a recursive algorithm.

We shall meet this technique here in the context of the MERGE SORT. (You should compare the MERGE SORT with the BUBBLE SORT, which we studied in Tutorial 1.)

We begin with the task of merging two lists which have already been sorted. For example the two lists:

<table>
<thead>
<tr>
<th>LIST 1</th>
<th>LIST 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chifley</td>
<td>Curtin</td>
</tr>
<tr>
<td>Fadden</td>
<td>Gorton</td>
</tr>
<tr>
<td>Forde</td>
<td>Hawke</td>
</tr>
<tr>
<td>Fraser</td>
<td>Howard</td>
</tr>
<tr>
<td>Holt</td>
<td>Menzies</td>
</tr>
<tr>
<td>Keating</td>
<td>Whitlam</td>
</tr>
</tbody>
</table>

Our algorithm for merging these two lists is as follows: At each step the elements at the top of each list are compared, and the one which is “smaller” (that is, first in the alphabetical ordering) is removed and put at the bottom of a combined list (which is empty to begin with). The process is continued until one list is empty. Then the remainder of the other list is put at the tail of the combined list.

So in the above example we begin by comparing “Chifley” and “Curtin”. “Chifley” is smaller, so it is removed and the combined list now contains “Chifley”. Next we compare “Curtin” and “Fadden”. “Curtin” is smaller, so it is removed and the combined list becomes:

**Combined List**

Chifley
Curtin

(i) How many comparisons are needed to merge LIST 1 and LIST 2?

(ii) More generally, if we are given two sorted lists, one with \( n \) elements and the other with \( m \) elements, prove that the total number of comparisons required to merge these two lists is not more than \( m + n - 1 \).
Suppose now that we are given the following list containing 12 members and asked to sort it:

Keating, Fadden, Fraser, Holt, Forde, Chifley, Hawke, Curtin, Whitlam, Gorton, Menzies, Howard

We adopt the divide and conquer technique. First, divide the group of 12 into two groups of 6. We shall sort each of these groups of 6, and then merge the two sorted groups.

How do we sort each group of 6? We divide it into 2 groups of 3, sort these and then merge.

How do we sort a group of 3? We divide it into two groups, one with 2 elements and the other with 1 element, sort the group with 2 elements and merge it with the group with 1 element.

How do we sort a group of 2 elements? Divide it into two groups with 1 element and then merge.

So now we know how to sort the list with 12 elements!!

(iii) Show that, using this procedure, any list with 3 elements can be sorted in not more than 3 comparisons.

(iv) Show that, using this procedure, any list with 6 elements can be sorted in not more than 11 comparisons.

(v) Show that, using this procedure, any list with 12 elements can be sorted in not more than 33 comparisons.

Recall that the BUBBLE SORT required $\frac{(n^2-n)}{2}$ comparisons in the worst case. So if $n = 12$, then 66 comparisons may be needed. The MERGE SORT, though not as easy to describe, is significantly better.

We shall now analyze the general case of MERGE SORT. To do this, we need some notation. Let $T(n)$ be the number of comparisons needed to sort a list with $n$ elements using the MERGE SORT.

(vi) Prove that $T(2^n) < 2T(2^{n-1}) + 2^n$, for any positive integer $n$.

[Hint: Consider a list with $2^n$ elements, and look at the first step of the MERGE SORT; that is, dividing the list into two groups each with $2^{n-1}$ elements.]

(vii) Using induction and (vi), prove that $T(2^n) < n.2^n$, for any positive integer $n$.

(viii) For any real number $a$, let $\lfloor a \rfloor$ denote the largest integer not greater than $a$. (e.g., $\lfloor 2.3 \rfloor = 2$, and $\lfloor -4.6 \rfloor = -5$.) Show that $a \leq 2^{\lfloor \log_{10}a \rfloor + 1}$, for any positive real number $a$.

(ix) Using (vii), (viii), and the fact that $m \leq k$ implies $T(m) \leq T(k)$, prove that $T(m) \leq 4m \log m$, for any integer $m \geq 2$. 
(x) Deduce that $T$ is $O(n \log n)$.

(xi)* Prove that $T$ is not $O(n)$.

(xii)* Prove that, in the notation of Tutorial 1, $B$ is not $O(n \log n)$.

In Tutorial 1 we saw that, in the worst case, the BUBBLE SORT is $O(n^2)$. We have now shown that the MERGE SORT is $O(n \log n)$, which is significantly better. Part (xii) shows that the BUBBLE SORT is not $O(n \log n)$.

Indeed it can be shown that no (sequential) sorting algorithm can do better than $O(n \log n)$ – but this does not mean that there are not better sorting algorithms.

We have ignored many important questions which are central to implementing a sorting algorithm. For example, how much storage is needed to implement either the MERGE SORT or the BUBBLE SORT. You should give some thought to what other features of a sorting algorithm should be considered when evaluating it.
Question 1

The INSERTION SORT is one of the simplest sorting algorithms. Consider the following:

<table>
<thead>
<tr>
<th>LIST 1</th>
<th>List 2</th>
<th>List 3</th>
<th>List 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Britain</td>
<td>Britain</td>
<td>Austria</td>
<td>Austria</td>
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<tr>
<td>Sweden</td>
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<td>Britain</td>
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<td>Austria</td>
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<tr>
<td>Spain</td>
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<td>Taiwan</td>
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<tr>
<td>Australia</td>
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<td>Australia</td>
<td>Australia</td>
</tr>
<tr>
<td>New Zealand</td>
<td>New Zealand</td>
<td>New Zealand</td>
<td>New Zealand</td>
</tr>
</tbody>
</table>

List 1 is the given list. At the $i^{th}$ step we regard the first $i$ elements as already sorted and put the $(i + 1)^{st}$ element in its correct position among the first $i$ elements. In the above example this is done as follows:

In each list we have sorted down to the element underlined. List 1 is the given list, so we regard “Britain” as being in its correct position. Now we look at the second member of the list, “Sweden”, and compare it with “Britain”. It is later in the alphabet, so in List 2 we put “Sweden” in second position, and we have sorted the first two elements. We have used 1 comparison.

Next we compare “Austria” with “Sweden”. It comes earlier in the alphabet. So we compare “Austria” with “Britain”. Once again it comes earlier in the alphabet, so “Austria” moves to the top of the list. Thus in List 3 we have sorted the first 3 members. This time we used 2 comparisons.

Next we compare “Hong Kong” with “Sweden”. It comes earlier in the alphabet, so we compare “Hong Kong” with “Britain”. It comes later in the alphabet. So in List 4 we have sorted the first 4 members, and used 2 more comparisons.
(i) Continue the insertion sort by writing down LIST 5, LIST 6, and LIST 7, recording at each stage the number of comparisons performed.

The INSERTION SORT has its worst case (that is the one requiring the most comparisons) when the list is in reverse alphabetical order. (You should NOT think that every sort has its worst case when the list is in reverse alphabetical order. Some sorts have their worst case when the given list is already in alphabetical order.)

Let $L_n = \{a_1, a_2, \ldots, a_n\}$ be a list of names in reverse alphabetical order. Let $I(n)$ be the number of comparisons required to put this list in alphabetical order.

(ii) Apply the INSERTION SORT and write down LIST 1, LIST 2, LIST 3, and LIST 4. Each time indicate how many comparisons are performed.

(iii) Prove that $I(n) = n(n - 1)/2$.

(iv) Prove that $I$ is $O(n^2)$.

**Question 2**

The SELECTION SORT is another very simple sorting algorithm. Consider the following:

<table>
<thead>
<tr>
<th>LIST 1</th>
<th>List 2</th>
<th>List 3</th>
<th>List 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Britain</td>
<td>Australia</td>
<td>Australia</td>
<td>Australia</td>
</tr>
<tr>
<td>Sweden</td>
<td>Sweden</td>
<td>Austria</td>
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<td>Austria</td>
<td>Austria</td>
<td>Sweden</td>
<td>Britain</td>
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<tr>
<td>Hong Kong</td>
<td>Hong Kong</td>
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<td>Thailand</td>
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<td>Australia</td>
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<td>Britain</td>
<td>Sweden</td>
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<tr>
<td>New Zealand</td>
<td>New Zealand</td>
<td>New Zealand</td>
<td>New Zealand</td>
</tr>
</tbody>
</table>

List 1 is the given list. First we find the element which is earliest in the alphabet, and interchange it with the element at the top of the list. It is found by comparing “Britain” and “Sweden”and saying “Britain” is earlier; so we compare “Britain” with “Austria”– - “Austria”
is earlier; next we compare “Austria” and “Hong Kong” — “Austria” is earlier; next we compare “Austria” and “Thailand” — “Austria” is earlier; we continue until we have compared “Austria” and “Australia”, and found that “Australia” is earlier. Then we compare “Australia” and “New Zealand” and find “Australia” is earlier. At this stage we know that “Australia” is first in the alphabetical ordering. So in LIST 2, “Australia” and “Britain” are interchanged. Next we find the second earliest element and exchange it with “Sweden”. We find the second earliest element by the same process, except we begin by comparing “Sweden” and “Austria”, and move down LIST 2. So in LIST 2 the top element is in its correct position. In LIST 3 the top 2 are in their correct position.

(i) How many comparisons were done in proceeding from LIST 1 to LIST 2?

(ii) How many comparisons were done in proceeding from LIST 2 to LIST 3?

(iii) How many comparisons were done in proceeding from LIST 3 to LIST 4?

(iv) How many comparisons are used in the entire sorting process.

(v) Have you noticed that the number of comparisons is INDEPENDENT of the order of the given list? Is this a good feature?

(vi) Let $S(n)$ be the number of comparisons needed to sort a list with $n$ elements by SELECTION SORT. Find a “closed form expression” for $S(n)$, and hence verify that $S$ is $O(n^2)$. 

Question 1

In previous tutorials we have studied BUBBLE SORT, MERGE SORT, INSERTION SORT, and SELECTION SORT. In this tutorial we study one of the best sorting algorithms, namely QUICKSORT. It is very fast – indeed, its average running time is less than that of all other known sorting algorithms.

The algorithm for QUICKSORT is a little difficult to describe, so we shall simplify the analysis by considering only the number of comparisons required, and ignoring the number of interchanges of elements needed. (If you study Computer Science, you will meet QUICKSORT in its full glory. For those who cannot wait, see “Data Structures and Algorithms” by A. Aho, J.E. Hopcroft, and J.D. Ullman [Addison-Wesley, 1983] and “Handbook of Algorithms and Data Structures” by G.H. Gonnet [Addison-Wesley, 1984].) So this tutorial should be regarded as an introduction to QUICKSORT.

In Tutorial 2 you met the technique called DIVIDE AND CONQUER – this is the method in which a problem is solved by dividing it into smaller subproblems, solving the subproblems, and then combining their solutions to produce the solution to the original problem. This technique is very useful when the subproblems are of the same type as the original problem so that a recursive approach is possible. This is the case for QUICKSORT.

The idea behind QUICKSORT is easy: to QUICKSORT a list \( L \), select an arbitrary element \( x \) of \( L \) and call this element the pivot. Then rearrange the list so that all of the entries that are smaller than \( x \) precede it, and all of the entries that are bigger than \( x \) follow it. As a result of this rearrangement \( x \) is in its correct position. Next let \( L_1 \) be the list of entries that are greater than \( x \) and \( L_2 \) the list of entries that are less than \( x \). Since \( x \) is in its correct position, all we need to do is to sort \( L_1 \) and \( L_2 \).

Thus we have reduced the problem of sorting the list \( L \) to that of sorting the smaller lists \( L_1 \) and \( L_2 \). That is we are dividing and conquering. So we repeat this procedure on \( L_1 \) and \( L_2 \), and proceed recursively.

We shall now look at an example. To make the analysis easier – rather than choosing a random element of the list as the pivot we shall always choose the first element of the list.
LIST 1 | LIST 2 | LIST 3 | LIST 4 \\
---|---|---|---
EASTERN SPINEBILL | CURRAWONG | COCKATOO | BUTCHERBIRD* \\
currawong | cockatoo | butcherbird | COCKATOO* \\
rosella | butcherbird | crow | CROW* \\
magpie | crow | CURRAWONG* | CURRAWONG* \\
wedge tailed eagle | eastern spinebill* | eastern spinebill* | eastern spinebill* \\
cockatoo | ROSELLA | MAGPIE | FAIRY WREN \\
wattlebird | magpie | fairy wren | GALAH \\
butterbird | wedge tailed eagle | kookaburra | MAGPIE* \\
crow | wattlebird | rosetta* | ROSELLA* \\
fairy wren | fairy wren | rosetta* | WATTLEBIRD* \\
galah | galah | wedge tailed eagle | WATTLEBIRD* \\
kookaburra | kookaburra | wattlebird | 

LIST 1 is the given list of common Australian birds. As indicated above, we choose the top element “eastern spinebill” as the pivot. (To indicate what we are doing, pivots are typed in upper case letters.) So we compare it with all other elements in LIST 1. There are 4 entries which come earlier in the alphabet. So “eastern spinebill” is put in position 5 in LIST 2 and the 4 smaller entries are put above it (in the same order as they were in LIST 1). The other 7 entries are put after “eastern spinebill” in the same order as they appeared in LIST 1. [The precise manner of implementing this on a computer is omitted so as to simplify our analysis.] (We use asterisks to indicate elements which we know are in their final position. So in LIST 2 “eastern spinebill” is followed by an asterisk.)

Next we sort LIST 2. BUT LIST 2 is really two lists, namely the list of 4 elements above “eastern spinebill” and the list of 7 elements under it. (“eastern spinebill” is already in its final position.) Our pivot in the top list of 4 elements is “currawong”. There are 3 entries in the top list which are smaller than it. So in LIST 3 “currawong” appears in position 4. The 3 smaller elements appear above it. The pivot for the bottom list of 7 elements in LIST 2 is “rosella”. Four of the entries below it are smaller than it. So in LIST 3 these 4 entries are above it; that is “rosella” appears in position 10, and the other 2 entries appear below it. So in LIST 3, “currawong”, “eastern spinebill”, and “rosella” are in their final positions.

Now we have 3 sublists of LIST 3 to sort. They have pivots “cockatoo”, “magpie”, and “wedge tailed eagle”… and so on.

(i) Complete the QUICKSORT of this example.

QUICKSORT, as described above, has its worst case when the given list is already in alphabetical order.
Let LIST 1 be the list $a_1, a_2, \ldots, a_n$ and assume that it is in alphabetical order. We now QUICKSORT this list.

(ii) Write down LIST 2 and LIST 3 which are obtained from this LIST 1 and indicate how many comparisons are used.

(iii) How many comparisons are used in sorting LIST 1 entirely using QUICKSORT?

We see from the answer to (iii) that, in the worst case, QUICKSORT uses $T(n)$ comparisons, where $T$ is $\Theta(n^2)$. BUT we shall show that on the average, QUICKSORT uses $Q(n)$ comparisons, where $Q$ is $O(n \log n)$. We now turn to the average case.

Our first task is to define what we mean by the average number of comparisons needed to sort a list with $n$ elements.

Let $S = \{b_1, b_2, \ldots, b_n\}$ be a set of $n$ words.

(iv) How many different orderings of this set are there?

We can use QUICKSORT to sort each of these ordered sets and note the number of comparisons used. We define $Q(n)$ to be the average of these numbers; that is, $Q(n)$ is obtained by adding the number of comparisons used in QUICKSORTING all these ordered sets and dividing by $n!$.

(v) Evaluate $Q(3)$ by considering the 3–element set $\{banana, pear, apple\}$.

(vi) Verify that $Q(n) \leq \frac{3}{2} n \log n$, for $n = 1, 2, 3$.

**Question 2**

QUICKSORT the following list into alphabetical order:

Joan, Gary, Chris, Sid, Norm, Gwenda .

**The rest of this tutorial is for OPTIONAL reading.**

Returning now to our set $S$ with $n$ elements, we see that the probability that the pivot $b_1$ is the $i^{th}$ smallest element is $\frac{1}{n}$, since all $n$ possibilities are equally likely. If $b_1$ is the $i^{th}$ smallest element then, after $n - 1$ comparisons, we know the $i - 1$ elements which are smaller than it and the $n - i$ elements which are bigger than it. Next we have to sort the set with $i - 1$ elements and the set with $n - i$ elements. Thus we obtain

$$Q(n) = n - 1 + \frac{1}{n} \sum_{i=1}^{n} [Q(i - 1) + Q(n - i)].$$

(a) Prove that $\sum_{i=1}^{n} Q(n - i) = \sum_{j=0}^{n-1} Q(j)$. 
(b) Prove that \( \sum_{i=1}^{n} Q(i - 1) = \sum_{j=0}^{n-1} Q(j) \).

(c) From (a) and (b) we obtain that

\[
Q(n) = n - 1 + 2 \frac{\sum_{j=0}^{n-1} Q(j)}{n}.
\]

Using this, evaluate \( Q(4) \) and \( Q(5) \).
TUTORIAL 5
Features of Some Graphs

Question 1
Without trying to find a Eulerian path, state which of the following graphs are Eulerian, which are not Eulerian but contain an Eulerian path, and which ones do not contain an Eulerian path.

![Graphs for Question 1](image)

Question 2
(i) For each graph (b), (c) and (f) in Question 1:
   (a) write down an ordered edge list, and
   (b) use Fleury’s algorithm to construct an Eulerian path.

(ii) Repeat (a) and (b) for the following graph

![Graph for Question 2](image)

Question 3
(i) Complete the first four columns of the following table for the examples below. See if you can guess some pattern based on the degree of the vertices.

<table>
<thead>
<tr>
<th></th>
<th>Number of edges</th>
<th>Number of vertices</th>
<th>Number of vertices of odd degree</th>
<th>Number of vertices of even degree</th>
<th>$N_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>2</td>
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<td>10</td>
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</tr>
</tbody>
</table>

(ii) For a graph $G$ with $V$ vertices of degree $d_1, d_2, \ldots, d_v$, define $N_G$ to be the sum of the degrees of all the vertices; i.e. $N_G = d_1 + d_2 + \cdots + d_v$. Use this definition to complete Column 5 in the table.

(iii) Guess a simple formula for $N_G$, then prove that your guess is correct.

(iv) Use (iii) to prove that there is an even number of vertices of odd degree. (This should have been your guess in (i).)
Question 4

A simple graph is one without multiple edges or loops. What is the minimum and maximum number of edges a simple graph with $V$ vertices can have?

Would your answer change if the graph is required to be connected?
Recall the algorithm, covered in the lectures notes, for finding a spanning tree for a connected graph with vertices \( v_1, v_2, \ldots, v_n \) and edges \( e_1, e_2, \ldots, e_m \).

(i) (a) For \( i = 1 \) to \( n \), let \( L(i) = i \). Let \( E = \emptyset \)

(ii) For \( r = 1 \) to \( m \), perform the following steps.

(a) If \( e_r = \{v_i, v_j\} \) with \( L(i) \neq L(j) \), then let \( E = E \cup \{e_r\} \), and let \( L(i), L(j) \), and any \( L(k) \) such that \( L(k) = L(i) \) or \( L(k) = L(j) \), be replaced by \( \min\{L(i), L(j)\} \).

(b) If \( e_r = \{v_i, v_j\} \) with \( L(i) = L(j) \) then leave \( E, L(i) \) and \( L(j) \) as they are.

**Question 1**

Use the algorithm above to find a spanning tree for the following graph. Set out your answer in a table, as in the lecture notes.

**Kruskal’s Algorithm**

Consider the problem of building a communications network linking six centres, where the cost of links is not constant.

The following is a graph representing the problem. The vertices represent the centres, and the edges represent the links connecting the centres. The number attached to each edge is the amount (in $1000’s) of the cost of building the link. This cost factor is called the weight of the edge.
Obviously not all of the links need be built, but which ones should be built so that each centre is able to communicate with any other centre, and the network can be built as cheaply as possible?

We are able to solve this problem by finding a spanning tree of *minimal weight* for the graph. We do this using a modification of the above algorithm.

Before implementing the algorithm, we order the edges in increasing weight. So, for instance, the edge list for the graph above could be

\[
\begin{array}{c|c}
  e_i & w(e_i) \\
  \hline
  e_1 = \{v_2, v_3\} & w(e_1) = 1 \\
  e_2 = \{v_2, v_6\} & w(e_2) = 1 \\
  e_3 = \{v_3, v_6\} & w(e_3) = 2 \\
  e_4 = \{v_1, v_2\} & w(e_4) = 3 \\
  e_5 = \{v_1, v_5\} & w(e_5) = 3 \\
  e_6 = \{v_3, v_4\} & w(e_6) = 4 \\
  e_7 = \{v_4, v_5\} & w(e_7) = 5 \\
  e_8 = \{v_5, v_6\} & w(e_8) = 6 \\
\end{array}
\]

We then consider each edge \( e_1, e_2, \ldots, e_8 \) in order.

An extra column is needed in the table for the weight tally at each step.

**Question 2**

Use Kruskal’s algorithm to find a minimal spanning tree for the graph.

**Question 3**

Use Kruskal’s algorithm to find minimal spanning trees for the following graphs.
Question 4

Let $G$ be a connected graph. What can you say about an edge of $G$ which appears in
(i) every spanning tree?
(ii) no spanning tree?
Justify your answers.
TUTORIAL 7
Boolean Algebra and Voting Machines

Question 1
Two terms in a Boolean expression are adjacent if they differ by exactly one literal, and this literal appears in one term, while its complement appears in the other.

e.g. $xyz$ and $xyz'$ are adjacent,

but $xyz$ and $x'yz'$ are not adjacent.

Write down all pairs of adjacent terms in the following Boolean expressions.

(i) $xyz + x'yz + x'y'z' + x'y'z'$
(ii) $wx'y'z + wxy'z + wx'y'z + w'xyz'$
(iii) $w'x'y'z + wxy'z + wxyz' + wxy'z' + wxyz$
(iv) $wxy'z + w'x'y'z + w'y'z + wy'z + w'xy'z + wxyz$

Question 2
Two adjacent terms in 1(i) are $xyz$ and $x'yz$. We can simplify the sum of these terms by using the properties of the Boolean algebra (see B1 – B5 of the 2nd Lecture). For example,

$$xyz + x'yz = (x + x')yz \quad \text{using (f)}$$
$$= lyz \quad \text{using (i)}$$
$$= yz \quad \text{using (h)}$$

Use the properties of the Boolean algebra to simplify the Boolean expressions given in Question 1.

Question 3

(i) Draw a switching circuit for the expression given for the Boolean function $f : S \times S \times S \rightarrow S$ given by $f(x, y, z) = xyz + xyz' + xy'z' + x'yz'$.

(ii) Simplify this expression for $f(x, y, z)$.

(iii) Draw a switching circuit for the simplified expression.
Question 4

Designing Electronic Voting Machines

Suppose there is a committee of four people, each with a switch to record their vote, say \( w, x, y \) and \( z \). The voting rights are weighted so that

- \( w \) has a block of 5 votes,
- \( x \) has a block of 4 votes, and
- \( y \) and \( z \) each have a block of 3 votes.

A motion will be passed if it gains at least half of the votes, and the switching circuit should be closed if the motion is passed.

(i) Complete the following voting table.

<table>
<thead>
<tr>
<th>Votes for</th>
<th>votes against</th>
<th>Circuit condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w ) ( x ) ( y ) ( z )</td>
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</table>

(ii) From the table, write down the corresponding Boolean expression.

(iii) Draw the switching circuit for this Boolean expression.

(iv) Simplify the Boolean expression found in (ii).

(v) Draw a simplified circuit for the electronic voting machine.
TUTORIAL 8
Logic Gates and Digital Adders

Computers contain silicon chips and printed on these chips are logic circuits. We will design ‘small’ logic circuits, a silicon chip may have thousands of such ‘small’ circuits. The basic building blocks of a logic circuit are called logic gates. These logic gates carry out the various logical operations on their inputs, we will define five of them.

\[ p \quad q \quad pq \quad p+q \quad p' \]
\[(a) \quad (b) \quad (c)\]

‘and’ gate (forms \( p \land q \) from \( p, q \)) (figure (a)), ‘or’ gate (forms \( p \lor q \) ) (figure (b)) ‘not’ or ‘inverter’ or ‘complement’ gate (forms \( \sim p \) ) (figure (c)).

‘nand’ gate (‘not and’) (figure (a)), equivalent to figure (b):

\[ p \quad q \quad (pq)' \quad p \quad (pq)' \]
\[(a) \quad (b)\]

‘nor’ gate (‘not or’) (figure (a)), equivalent to figure (b):

\[ p \quad q \quad (p+q)' \quad p \quad (p+q)' \]
\[(a) \quad (b)\]

To calculate the output from a logic circuit or to design a logic circuit implementing a given logical proposition we use the machinery of Boolean algebra.

Examples

(i). Calculate the output from
**Solution:** We simply work our way from the left (input) to the right (output) putting in the various outputs according to (a) to (e) above.

\[
\begin{array}{c}
 x \\
 y
\end{array} \xrightarrow{\text{NOT}} \begin{array}{c}
 x' \\
 y
\end{array} \xrightarrow{\text{AND} \atop \text{AND}} \begin{array}{c}
 x'+y \\
 x
\end{array} \xrightarrow{\text{AND} \atop \text{AND}} (x'+y)x
\]

the output is

\[
(x'+y)x = x'x + xy = xy \text{ (by axiom (j))}.
\]

(ii). Give the gate implementation of \(xy + x'\)

**Solution:** We work backwards from the output:

Firstly,

\[
\begin{array}{c}
 x' \\
 xy
\end{array} \xrightarrow{\text{OR} \atop \text{AND}} \begin{array}{c}
 xy+x' \\
 xy
\end{array}
\]

Now \(x'\) is produced by

\[
\begin{array}{c}
 x
\end{array} \xrightarrow{\text{NOT}} x'
\]

and \(xy\) by

\[
\begin{array}{c}
 x \\
 y
\end{array} \xrightarrow{\text{AND} \atop \text{AND}} xy
\]

Putting this together we have

\[
\begin{array}{c}
 x \\
 y
\end{array} \xrightarrow{\text{AND} \atop \text{AND}} \begin{array}{c}
 xy
\end{array} \xrightarrow{\text{OR} \atop \text{AND}} \begin{array}{c}
 x' \\
 xy
\end{array} \xrightarrow{\text{OR} \atop \text{AND}} xy+x'
\]
(iii). Give the gate implementation of \(xyz' + z\).

**Solution:**

\[
\begin{array}{c}
\text{x} \\
\text{xy} \\
\text{y} \\
\text{z'} \\
\text{z} \\
\text{xyz'} \\
\text{xyz'}+z
\end{array}
\]

**Question 1**

Write down minimal expressions for the following functions, and give the gate implementation for the outputs.

(a) \(x y\) output
(b) \(x y z\) output
(c) \(x y z\) output

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>output</th>
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<tr>
<td>0</td>
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<th>x</th>
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**Question 2**

**Half Adders** Consider adding two binary digits \(x\) and \(y\) (each could be 0 or 1) and expressing their sum as a two-digit binary number \((cs)_2\).

(a) Fill in the following tables for \(s\) and \(c\).

(b) Give the gate implementations for the outputs \(s\) and \(c\).

(c) Give one gate implementation that inputs \(x\) and \(y\) and outputs \(s\) and \(c\).

The circuit you have just drawn is called a half-adder and is denoted by
Question 3

Full Adders Suppose you wanted to design a logic circuit for the addition of three binary digits \(x, y\) and \(z\). Let \(x + y + z = (cs)_2\).

(a) Fill in the following table.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
<th>(c)</th>
<th>(s)</th>
</tr>
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<tbody>
<tr>
<td>0</td>
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(b) Using half adders, give the gate implementation for the circuit that inputs \(x, y\) and \(z\), out outputs \(c\) and \(s\).

The circuit you have just drawn is called a full adder and is denoted by

\[
\begin{array}{c}
\text{FA} \\
\hline
x'yz' + x'yz' + xy'z' + xyz \\
xy + xz + yz
\end{array}
\]

*(c) Full adders are used in performing the operation of adding 2 multiple-digit binary numbers. Give the gate implementation for the addition of two two-digit binary numbers.

Question 4

(a) Give the gate implementation of the ‘nand’ gate using only ‘and’, ‘or’ and ‘not’ gates.

(b) Give the gate implementation of the ‘nor’ gate using only ‘and’, ‘or’ and ‘not’ gates.
Question 5

Give the gate implementation of the Boolean expression $y + (xz)'$ using

(a) only nand gates;
(b) only nor gates.
TUTORIAL 9
More on Karnaugh Maps

In general, when we use Karnaugh maps, we will want to

(a) use the least number of adjacency circles to cover all the 1’s (to minimize the number of product terms);
(b) include each 1 in the largest possible adjacency circle (to minimize the number of literals);
(c) Condition (a) takes precedence over (b).

These conditions are satisfied if we follow the algorithm of in the lecture notes.

Question 1
Use a Karnaugh map to find a minimal representation for the Boolean expression

\[ w'xyz + wx'y'z' + wx'y'z + w'x'y'z' + w'x'y'z' + wx'y'z' + wx'y'z' + w'xy'z' + wx'y'z \]

Question 2
Use a Karnaugh map to find a minimal representation for the Boolean expression

\[ wxyz + wx'y'z + wxyz' + wx'y'z' + wx'y'z' + w'xy'z' + wx'y'z' + wx'y'z + wx'y'z + w'xy'z \]

Question 3
(a) Use a Karnaugh map to find two minimal representations for the Boolean expression

\[ w'xyz + wx'y'z' + wx'y'z' + w'xy'z + wx'y'z + w'xy'z \]

(b) Explain why these equivalent expression are both minimal representations.

Question 4
A 4 × 8 grid is required when using a Karnaugh map to minimize a Boolean expression in 5 variables.

(a) Complete the labeling of the Karnaugh map.
Check that (a) is correct before going on!!

(b) The double line is a line of symmetry. Mark all boxes adjacent to Y.

(c) Use a Karnaugh map to minimize the following Boolean expression:

(i)

\[ vwxyz + vwxyz' + v'w'x'y'z + v'w'x'y'z' + vw'x'y'z' + vw'x'y'z' + v'w'x'y'z' + v'w'x'y'z' + v'w'x'y'z' + v'w'x'y'z' \]

\[ + v'w'x'y'z' + vw'x'y'z + vw'x'y'z + v'w'x'y'z + v'w'x'y'z + v'w'x'y'z + v'w'x'y'z + v'w'x'y'z \]

(ii)

\[ vw'x'yz + vw'x'yz + v'w'x'y'z + v'w'x'y'z + vwxyz' + vw'x'y'z' + v'wxy'z' \]

\[ + vwxy'z' + vw'x'y'z' + v'w'x'y'z' + v'wxy'z' + v'w'x'y'z \]
PRACTICE CLASS 1
Sets and Mathematical Induction

Question 1
Let $A$, $B$ and $C$ be subsets of set $U$. Show

a) $A - B = A \cap B'$.

b) $(A \cup B) - (C - A) = (A \cup B) \cap (C' \cup A)$,
   i.e. represent the l.h.s. without the use of the difference “−”.

c) $(A \cup B) - (C - A) = A \cup (B - C)$.

Question 2
Let $A$ be a set with two elements, say $A = \{1, 2\}$.

a) Give the power set $\mathcal{P}(A)$ of $A$.

b) Give the Cartesian product $A \times A$.

c) Give the Cartesian product $\mathcal{P}(A) \times A$.

Question 3
Write the following in $\sum$-notation:

a) $1 + 2 + \cdots + 10$

b) $1 + 3 + \cdots + (2n - 1)$

c) $1^3 + 2^3 + \cdots + k^3$

d) $4^2 + 6^2 + \cdots + k^2$

Question 4
Evaluate the following sums,

a) $\sum_{k=3}^{6} k^2$

b) $\sum_{n=2}^{4} \frac{1}{n}$

c) $\sum_{i=0}^{3} 3^i$

d) $\sum_{m=2}^{5} \frac{m}{2}$

Question 5
Prove by induction that $1 + 3 + 9 + \cdots + 3^{n-1} = \frac{3^n-1}{2}$ for each positive integer $n$. 
Question 6
Prove by induction that \( \sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2 \), for all integers \( n \geq 1 \).

Question 7
Prove by induction that \( \sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1} \), for all integers \( n \geq 1 \).

Question 8
Prove by induction that \( 2n + 1 < 2^n \), for all integers \( n \geq 3 \).

Question 9
Use the result in Question 8 and induction to prove that \( n^2 < 2^n \), for all integers \( n \geq 5 \).

Question 10
Prove by induction that \( 4^n - 1 \) is divisible by 3, for integers \( n \geq 1 \).

Question 11
Prove by induction that \( n^3 - n \) is divisible by 6, for all integers \( n \geq 1 \).

Question 12
Prove by induction that for every \( n \in \mathbb{N} \), \( 3^n + 7^n - 2 \) is divisible by 8.
PRACTICE CLASS 2
Horner’s Algorithm, $O$ the $\Theta$ Notation

Question 1
i) Write $f(x) = 4x^3 - 3x^2 + x - 2$ in telescoping form.
ii) Evaluate $f(1)$ and $f(-1)$ by Horner’s method.

Question 2
(i) Write $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ in telescoping form.
(ii) Let $c$ be any real number. How many multiplications are used in evaluating $f(c)$ when $f(x)$ is written in telescoping form.
(iii) How many multiplications are used in evaluating $f(c)$ from the given form of $f(x)$ rather than the telescoping form?
(iv) How many additions/subtractions are used in evaluating $f(c)$ when $f(x)$ is written in telescoping form?
(v) How many additions/subtractions are used in evaluating $f(c)$ from the given form of $f(x)$ rather than the telescoping form?
(vi) Is it clear from the above calculations why Horner’s method is used in preference to the standard substitution method?

Question 3
Let $f(x) = 2x^3 + 3x^2 + x - 1$ for $x \in \mathbb{R}$. Prove $f(x) = O(x^3)$.

Question 4
Let $\log_2$ be denoted simply by $\log$, as is often the case in the literature of computer science. For simplicity, we allow $n$ below only to be positive integers.

(i) First, prove that $n \leq 2^n$, for each $n \in \mathbb{N}$. Then prove $\log n \leq n$.
(ii) Write down the definitions of $\log n$ being $O(n)$, i.e. $\log n = O(n)$.
(iii) Prove, using (i), that $\log n = O(n)$.
(iv) $1$ can be regarded as the function $g: \mathbb{N} \rightarrow \mathbb{R}$ with $g(n) = 1$ for all $n \in \mathbb{N}$. Prove that $1 = O(\log n)$.
(v) Prove that $n = O(n \log n)$.
(vi) Prove that $n \log n = O(n^2)$. 
(vii) Prove that $2^n < n!$, for all $n \geq 4$.

(viii) Using (vii) prove that $2^n = O(n!)$.

*(ix) Prove $n^2 = O(2^n)$.

Hence we have shown in this question

\[ 1 = O(\log n), \quad \log n = O(n), \quad n = O(n \log n), \]
\[ n \log n = O(n^2), \quad n^2 = O(2^n), \quad 2^n = O(n!). \]

Question 5

Let $f(n) = n^3 - 4n^2 - n + 1$ for $n \in \mathbb{N}$. Prove $f(n) = \Theta(n^3)$.

Question 6

Let $f(n) = \frac{4n^2 + 2 \log_2 n}{n}$ for $n = 1, 2, 3, \ldots$. Prove $f(n) = \Theta(n)$.

Question 7

Define $f : \mathbb{N} \to \mathbb{R}$ by $f(n) = n^3 - 2n^2 + \log n - 3$. Prove $f(n) = \Theta(n^3)$.
PRACTICE CLASS 3
Symbolic Logic and Logical Equivalence

Question 1
Let $p$ be ‘Mark is rich’ and $q$ be ‘Mark is happy’. Write each of the following in symbolic form. (Assume ‘poor’ and ‘not rich’ are the same!)

(i) Mark is poor and happy.
(ii) Mark is neither poor nor happy.
(iii) Mark is poor or else he is both rich and happy.

Question 2
In each case decide whether the proposition is True or False.

(i) $(3 + 7 > 10)$ or $(6 < 10)$.
(ii) $\sim (3 + 7 > 10)$.
(iii) $\sim ((2 + 2 = 3) \land (2 + 3 = 15))$.
(iv) If $1 + 2 = 4$ then $16 = 35$.
(v) If $x$ is a positive integer and $x^2 \leq 3$ then $x = 1$.

Question 3

(i) Write down a truth table to show $p \rightarrow q$ is equivalent to $\sim q \rightarrow \sim p$ and to $\sim p \lor q$.
(ii) Use (i) to write expressions using only $\land$, $\lor$ and $\sim$ that are equivalent to:
   (a) $(p \land q) \rightarrow r$; (b) $p \rightarrow (p \land q)$.

Question 4

(i) Write down a truth table to show that the negation of $p \rightarrow q$ is equivalent to $p \land (\sim q)$.
(ii) To show $p \rightarrow q$ is false, we must show $p$ is . . . and $q$ is . . . .

Question 5
Use de Morgan’s Laws to simplify the following:

(i) $\sim (p \land q)$
(ii) $\sim (\sim p \lor q)$
(iii) $\sim (\sim p \land \sim q)$
(iv) $\sim ((p \land q) \lor \sim q)$.
PRACTICE CLASS 4
Propositions and Predicates

Question 1
Use a truth table to show that

\[ \sim p \land q, \sim p \rightarrow r, \sim q \rightarrow r, \therefore r \]

is a valid argument.

Question 2
For the following valid argument form

\[ s \rightarrow \sim p, \ t \lor s, \sim s \rightarrow u \land q, \ p, \ t \land q \rightarrow r, \therefore r \]

list the valid elementary argument forms or inference rules that are used to derive the conclusion.

Question 3
For the following valid argument form

\[ \sim p \lor q \rightarrow r, \ s \lor \sim q, \sim t, \ p \rightarrow t, \sim p \land r \rightarrow \sim s, \therefore \sim q \]

list the valid elementary argument forms or inference rules that are used to derive the conclusion.

Question 4
Show that the following propositions

1. \[ P \rightarrow (Q \rightarrow P) \]
2. \[ (\sim Q \rightarrow \sim P) \rightarrow (P \rightarrow Q) \]
3. \[ (\forall x, P(x) \rightarrow Q(x)) \rightarrow ((\forall x, P(x)) \rightarrow (\forall x, Q(x))) \]

are all tautologies.

Question 5
Indicate whether the arguments in (i) and (ii) below are valid or invalid. Support your answers by drawing set diagrams.

(i) All people are mice.
All mice are mortal.
All people are mortal.

All teachers occasionally make mistakes.

No gods ever make mistake.

∴ No teachers are gods.

**Question 6**

Reorder the premises in each of the arguments so that the conclusion follows logically.

1. I trust every animal that belongs to me.
2. Dogs gnaw bones.
3. I admit no animals into my study unless they will beg when I tell them to do so.
4. All animals in the yard are mine.
5. I admit every animal that I trust into my study.
6. The only animals that are really willing to beg when told to do so are dogs.

∴ All the animals in the yard gnaw bones.
PRACTICE CLASS 5
Isomorphism and Adjacency Matrix

Question 1
For each of the following pairs of graphs, find all the possible isomorphisms from graph (a) onto graph (b).

Question 2
Another way of describing a graph is to use an $n \times n$ matrix, an adjacency matrix. The entry in the $ij$-th position in the matrix is the number of edges connecting the $i$th and $j$th vertices. This method of describing a graph is most useful for storing the information in a computer. For example, the following gives two descriptions of the same graph.

\[
\begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{pmatrix}
\]

(i) Notice that the matrix is symmetrical about the leading diagonal. Why?
(ii) Why does the leading diagonal contain only zeros?
(iii) Write the corresponding adjacency matrix for the following graph.
**Question 3**
Recall that a complete graph $K_n$ has $n$ vertices, and each pair of vertices are joined by an edge. For which values of $n$ is the graph $K_n$ planar? Justify your answer.

**Question 4**
Draw three non-isomorphic planar graphs with 4 edges and 5 vertices and prove that they are not isomorphic.

**Question 5**
Find all non-isomorphic, non-planar graphs with 6 vertices and containing no loops or multiple edges. Explain why you have all possible graphs.
A Binary Tree is a special sort of tree, which can be built up from a single vertex, called the root, by successfully adding either a left edge or a right edge, or both.

Examples

In Example (i), the root has a left edge, but no right edge, and in Example (ii), the root has a right edge but no left edge. We consider these two trees to be different.

**Question 1**

The number of different binary trees with \( n \) vertices is \({2n \choose n}/(n+1) = \frac{(2n)!}{(n+1)n!}.\)

Verify this formula for \( n = 2 \) and \( n = 3 \) by drawing all binary trees with 2 or 3 vertices.

**Question 2**

Binary trees can be used to handle binary operations. We will look at how they can be used to store algebraic expressions involving the usual operations — addition, multiplication, division and subtraction. Consider, for example the expression

\[
\left(\frac{a + b}{\left(\frac{a - b}{2}\right)}\right) - \left(\frac{a}{b} - \left(\frac{b}{a}\right)\right)
\]

As a binary tree, we can describe this as

\[
-\quad L \quad R
\]

where \( L = \frac{a + b}{\left(\frac{a - b}{2}\right)} \) and \( R = \frac{a}{b} - \left(\frac{b}{a}\right) \). Then we treat \( L \) and \( R \) as the binary trees; that is,
Continuing this process, we obtain the binary tree

(a) Draw a binary tree to describe each of the following algebraic expressions.
(i) \(((a + b)/(c + d)) + a) - (((c + d)/(a + b)) + b);
(ii) \(((a * b) + (b/a))/(a/b) + b).

We must also be able to retrieve the algebraic expression described in a binary tree. Take the example before part (a). If we numbered each pronumeral and binary operation in the order they appear in the expression, then they appear in the binary tree in the following order.
Notice that we don’t get to the root 10 until we have been to every vertex in the left subtree (which has root 4). We don’t get to the root 4 until we have been to every vertex in the left subtree (which has root 2).

When we have reached the root of a tree, we then visit the right subtree.

This can be described as follows:

<table>
<thead>
<tr>
<th>Traversing Binary Trees</th>
</tr>
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<tbody>
<tr>
<td>(I) Traverse the left subtree.</td>
</tr>
<tr>
<td>(II) Visit the root.</td>
</tr>
<tr>
<td>(III) Traverse the right subtree.</td>
</tr>
</tbody>
</table>

There are three commonly used methods for traversing binary trees – ‘preorder traversal’, ‘postorder traversal’ and ‘inorder traversal’. The one described here is **inorder traversal**. Some calculators (for example, Hewlett-Packard) use postorder traversal.

So for example, the traversal list for the following binary tree is B, A, C

![Binary Tree](image)

(b) For each of the following binary trees, describe the order in which the vertices are traversed.

(i)  ![Binary Tree](image)

(ii)  ![Binary Tree](image)

(iii)  ![Binary Tree](image)
Question 3  Sorting using Binary Trees

Suppose we wanted to sort a list of numbers into increasing order. We firstly create a binary tree containing these numbers. Then we use the method given above to create a traversal list.

Creating Binary Trees  Insert the first number into the root. We then insert each new number \( n \) in turn by:

- (A) moving down the left subtree if \( n \) is less than the number already at any vertex in the subtree;
- (B) moving down the right subtree if \( n \) is greater than or equal to the number already at any vertex in the subtree; and
- (C) creating a new vertex to hold the number \( n \) when the procedure in (A) and (B) would require moving down an ‘empty’ subtree.

(a) Create a binary tree containing the list 25, 17, 9, 20.
(b) Create a binary tree containing the list 25, 17, 9, 20, 13, 35, 30.
(c) Traverse the trees in (a) and (b).

(i) For each of the 6 possible listings of the numbers 1, 2, 3, create a binary tree to contain that list, noting the number of comparisons required.
(ii) For a list with 3 elements what is the average number of comparisons required?

In fact it can be shown that the average number of comparisons using this sorting technique has complexity

\[ \Theta(n \log n) \]

(d) Use a binary tree to sort the following lists of numbers, noting the number of comparisons required in each case.

(i) 1,2,3,4,5,6,7.
(ii) 2,4,6,1,3,5,7.
(iii) 4,2,6,1,3,5,7.

(e) Suppose you were given a list containing the first 15 numbers to sort.

(i) What orderings of this list would require the most number of comparisons?
(ii) What orderings of this list would require the least number of comparisons?

*(f) Repeat (e) for a list containing the first \( n \) numbers.
PRACTICE CLASS 7

Number Bases

Question 1 Use the Division Algorithm to convert the decimal expressions:
   a) 115 to base 3  b) 115 to base 5  c) 206 to base 5  
   d) 12 to base 3  e) 67 to base 2  f) 67 to base 4  
   g)* .12 to base 5  h)* .6 to base 3  i)* .56 to base 4

Question 2 Convert the following expression in base 8 to base 2.
   a) 34  b) 21  c) 74  
   d) 265  e) 752  f) 5724

Question 3 Do the following additions in binary arithmetic.
   a) 10110  b) 101  c) 1101101
       +11011  +1111  + 101011
   d) 1101  e) 1011  f) 10101
       10000  11001  1101
       + 1100  + 1101  11
       + 10111  + 101

Question 4 Do the following subtractions in binary arithmetic.
   a) 10011  b) 10111  c) 1101111
       − 1101  − 1110  − 10001
   d) 1011100  e) 100111  f) 1100011
       − 111100  −100101  − 101111

Question 5 Do the following multiplications and divisions.
   a) 10111 × 1111  b) 10100 × 111
   c) 11011 ÷ 11  d) 1101100 ÷ 1001
PRACTICE CLASS 8
Equivalence Relations and Partial Order Relations

Question 1
Let $\mathbb{R}$ be the set of real numbers, we define a relation $R$ on $\mathbb{R}$ by

$$\forall x, y \in \mathbb{R} : \quad x R y \quad \text{iff} \quad |x - y| \leq 1 .$$

(i) Draw the relation set $R$ explicitly on the plane $\mathbb{R} \times \mathbb{R}$.
(ii) Is the relation reflexive?
(iii) Is the relation symmetric?
(iv) Is the relation an equivalence relation?
(v) Is the relation an partial order relation?

Question 2
Let $A$ be any subset of $\mathbb{N}$ and is composed of some integers $\geq 1$. We define a relation $R \subseteq A \times A$ by

$$\forall m, n \in A : \quad (m, n) \in R \quad \text{iff} \quad m | n .$$

Then show

(i) The relation $R$ is a partial order relation.
(ii) Suppose $A$ is given by $A = \{1, 2, 3, 9, 18\}$ and $a \preceq b$ iff $a | b$, then the relation defined by $\preceq$ is just a special case of (i) and is thus a partial order relation. List the ordered pairs in $R$ and draw the associated Hasse diagram.

Question 3
Let $A$ be the set of all integers greater than or equal to 2, and relation $R$ be defined by

$$\forall m, n \in A : \quad (m, n) \in R \quad \text{iff} \quad m | n .$$

We furthermore denote $R$ simply by $\preceq$. Show

(i) 2, 3 and 5 are all minimal elements of $A$.
(ii) 4 is not a minimal element of $A$.
(iii) $A$ does not have any maximal elements.
(iv) 2 is not a least element of $A$.
(v) For any integer $p \geq 2$, $p$ is a prime number if and only if $p$ is a minimal element of $A$. 

PRACTICE CLASS 9
Switching Circuits and Their Design

Question 1
Simplify, where possible, each of the circuits. (You may need to write down the corresponding Boolean expression, simplify, then draw the simplified circuit.)

(a) $a \land b'$
(b) $a \lor a'$
(c) $a \land c$
(d) $a \land b$
(e) $a' \land b' \land c$
(f) $a \land b' \land c'$

Question 2
Suppose we have a garage light that we wish to be able to control from either inside the house or the garage. So, the light will have two switches, and it must be possible to turn the light on or off from either switch.

(a) construct a table showing the possible positions (on or off) of the switches, and the corresponding outcomes.

(b) Write down a Boolean expression for the outcome.

(c) Draw a switching circuit for this situation.
PRACTICE CLASS 10
Solutions of Homogeneous Recurrence Relations

Question 1
Consider the following recurrence relations (difference equations) and initial conditions:

(a) \( S(k + 2) = 5S(k + 1) - 4S(k), \) for \( k \geq 0, \) \( S(0) = 2, \) and \( S(1) = 5. \)

(b) \( F(n + 1) = 5F(n) + 5^n, \) for \( n \geq 0, \) and \( F(0) = 3. \)

(c) \( R(n + 3) - 6R(n + 2) + 11R(n + 1) - 6R(n) = 0, \) for \( n \geq 0, \) \( R(0) = 3, \) \( R(1) = 6, \) and \( R(2) = 14. \)

(d) \( F_{k+2} = F_{k+1} + F_k, \) for \( k \geq 0, \) \( F_0 = 1 \) and \( F_1 = 1. \)

(i) Write down the order of each of the above recurrence relations.

(ii) State which of the recurrence relations are homogeneous.

(iii) Write down the characteristic equation of each of the homogeneous relations.

(iv) Solve each of the homogeneous recurrence relations.

Question 2
Solve the recurrence relations

(a) \( f(n + 2) + 6f(n + 1) + 9f(n) = 0, \) for \( n \geq 2, \) \( f(0) = 1, \) \( f(1) = 0. \)

(b) \( A_{n+3} - 3A_{n+2} + 4A_n = 0. \)
PRACTICE CLASS 11
Solutions of Recurrence Relations and Practical Problems

Question 1
Solve the following recurrence relations:

(a) \( S(k + 2) - 6S(k + 1) + 9S(k) = 2, \quad \text{for } k \geq 0 \)
(b) \( T(n + 1) - 5T(n) = 7^n, \quad \text{for } n \geq 0 \quad T(0) = \frac{9}{2}. \)
(c) \( P(n + 2) - 4P(n + 1) + 4P(n) = n^2, \quad \text{for } n \geq 0. \)

In “real life” the problems you meet are expressed in words and you have to convert the words to symbols. Only then can you manipulate those symbols to get a solution. Questions 2 and 3 below are expressed in words. Do TRY to convert the problems into symbols WITHOUT seeking assistance.

Question 2
A particle is moving in the horizontal direction. The distance it travels in each second is equal to twice the distance it travelled in the previous one. Let \( a_r \) denote the position of the particle in the \( r^{th} \) second. Determine \( a_r \) given that \( a_0 = 3 \) and \( a_1 = 10. \)

Question 3
You open a savings account which pays an annual interest rate of 10%. You deposit $100 when you open the account and you double your deposit each year. Let \( B(n) \) be the balance after \( n \) years. Find a “closed form expression” for \( B(n). \)
Question 1

Consider the following diagram of a river with two islands and 15 bridges. Is it possible to design a walk so that you traverse each bridge exactly once? Justify your answer.

Question 2

What is the algebraic expression held in the following binary tree?

Question 3

Solve the following recurrence relation.

\[ 6f(n + 2) - 5f(n + 1) + f(n) = 0, \ f(0) = 1, \ f(1) = 0. \]
**Question 4**

Use Kruskal’s algorithm to find a minimal spanning tree for the following weighted graph.

![Graph Image](image-url)

**Question 5**

Use the characteristic equation method to solve

\[ S(n + 2) - 2S(n + 1) + S(n) = 4, \ S(0) = 2, \ S(1) = 2. \]

**Question 6**

What are the outputs of the following logic circuit? Show each intermediate value.

![Logic Circuit Image](image-url)

**Question 7**

Simplify then draw a switching circuit for the Boolean expression \((x + y')(x + y'z') + xyz\).

**Question 8**

Let \( f : \mathbb{N} \to \mathbb{R} \) be given by \( f(n) = 2n^3 + 3n^2 \log n + 2n - 1 \). Prove that \( f(n) = \Theta(n^3) \).
APPLIED MATHEMATICS 140

Tutorials and Practice Classes

SOLUTIONS
Tutorial 1 Solutions

Question 1.

List after Pass 3
(4 exchanges done)
Amelia
Anita
Deborah
Kris
Maria
Louise
Nicole
Paula
Sandra

List after Pass 4
(1 exchange done)
Amelia
Anita
Deborah
Kris
Louise
Nicole
Paula
Sandra

(i) Kris
Maria
Louise
Nicole
Paula
Sandra

(ii) \( a_2, a_3, \ldots, a_n, a_1 \)

(iii) \( n - 1 \)

(iv) \( B(n) = B(n - 1) + n - 1 \)

(v) \[
\begin{align*}
B(1) &= 0, B(2) = B(1) + 1 = 1, B(3) = B(2) + 2 = 3, B(4) = B(3) + 3 = 6, \\
B(5) &= B(4) + 4 = 10, B(6) = B(5) + 5 = 15, B(7) = B(6) + 6 = 21, \\
B(8) &= B(7) + 7 = 28, B(9) = B(8) + 8 = 36, B(10) = B(9) + 9 = 45.
\end{align*}
\]

(vi) Let \( S_n \) be the statement “\( B(n) = \frac{n(n-1)}{2} \)” for each \( n \in \mathbb{N} \).

Then \( S_1 \) is the statement “\( B(1) = \frac{1(1-1)}{2} = 0 \)”, which is true.

Assume \( S_k \) is true; i.e., \( B(k) = \frac{k(k-1)}{2} \). Then

\[
B(k + 1) = B(k) + k = \frac{k(k-1)}{2} + k \quad \text{by the inductive assumption},
\]

\[
= \frac{k^2 + k}{2} = \frac{(k + 1)k}{2}.
\]

Therefore, by the Principle of Mathematical Induction, \( B(n) = \frac{n(n-1)}{2} \), for each \( n \in \mathbb{N} \).

(vii) \( B(10) = \frac{10(10-1)}{2} = 45 \)
Question 1.

(i) 10

(ii) Each time we compare two elements, one is removed and put in the combined list. If we have only 1 element left, no further comparisons are possible. So the number of comparisons is certainly no bigger than one less than the total number of elements; that is, \( \leq m + n - 1 \).

(vi) The list with \( 2^n \) elements is divided into two groups of \( 2^{n-1} \) elements. Each group of \( 2^{n-1} \) elements can be sorted with \( T(2^{n-1}) \) comparisons. These two sorted groups of \( 2^{n-1} \) elements can be merged in not more than \( 2^{n-1} \) comparisons. So

\[
T(2^n) \leq 2T(2^{n-1}) + 2^n - 1 < 2T(2^{n-1}) + 2^n.
\]

(vii) Let \( S_n \) be the statement “\( T(2^n) < n2^n \).” When \( n = 1 \), \( T(2^1) = T(2) = 1 \), and \( n2^2 = 2 \). So \( S_1 \) is true.

Assume \( T(2^k) < k2^k \) for some \( k \in N \). [We are required to prove \( T(2^{k+1}) < (k + 1)2^{k+1} \).]

\[
T(2^{k+1}) \leq 2T(2^k) + 2^{k+1} \quad \text{(by vii)}
\]
\[
< 2k2^k + 2^{k+1} \quad \text{(by assumption)}
\]
\[
= (k + 1)2^{k+1}
\]

(x) This follows from (ix), with \( c = 4 \) and \( M = 2 \).
Tutorial 3 Solutions

Question 1.

(i)

<table>
<thead>
<tr>
<th>List 5</th>
<th>List 6</th>
<th>List 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1 comparison)</td>
<td>(3 comparisons)</td>
<td>(5 comparisons)</td>
</tr>
<tr>
<td>Austria</td>
<td>Austria</td>
<td>Austria</td>
</tr>
<tr>
<td>Britain</td>
<td>Britain</td>
<td>Britain</td>
</tr>
<tr>
<td>Hong Kong</td>
<td>Hong Kong</td>
<td>Canada</td>
</tr>
<tr>
<td>Sweden</td>
<td>Norway</td>
<td>Hong Kong</td>
</tr>
<tr>
<td>Thailand</td>
<td>Sweden</td>
<td>Norway</td>
</tr>
<tr>
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<td>Sweden</td>
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<td>Canada</td>
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<td>Thailand</td>
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<td>United States</td>
<td>United States</td>
</tr>
<tr>
<td>Spain</td>
<td>Spain</td>
<td>Spain</td>
</tr>
<tr>
<td>Taiwan</td>
<td>Taiwan</td>
<td>Taiwan</td>
</tr>
<tr>
<td>Singapore</td>
<td>Singapore</td>
<td>Singapore</td>
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<tr>
<td>India</td>
<td>India</td>
<td>India</td>
</tr>
<tr>
<td>Australia</td>
<td>Australia</td>
<td>Australia</td>
</tr>
<tr>
<td>New Zealand</td>
<td>New Zealand</td>
<td>New Zealand</td>
</tr>
</tbody>
</table>

(ii)

<table>
<thead>
<tr>
<th>List 1</th>
<th>List 2</th>
<th>List 3</th>
<th>List 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1 comparison)</td>
<td>(2 comparison)</td>
<td>(3 comparisons)</td>
<td></td>
</tr>
<tr>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$a_3$</td>
<td>$a_4$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$a_3$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$a_3$</td>
<td>$a_1$</td>
<td>$a_2$</td>
</tr>
<tr>
<td>$a_4$</td>
<td>$a_4$</td>
<td>$a_4$</td>
<td>$a_1$</td>
</tr>
<tr>
<td>$a_5$</td>
<td>$a_5$</td>
<td>$a_5$</td>
<td>$a_5$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$a_n$</td>
<td>$a_n$</td>
<td>$a_n$</td>
<td>$a_n$</td>
</tr>
</tbody>
</table>
(iii) As 1 more element is sorted each time, the process will end with LIST \( n \). Since LIST \( i \) adds \( i - 1 \) comparisons we obtain:

\[
I(n) = 1 + 2 + 3 + \cdots + n - 1 = \frac{n(n - 1)}{2}
\]

(iv)

\[
I(n) = \frac{2}{2} \frac{n(n - 1)}{2} = \frac{1}{2} n^2 - \frac{1}{2} n \\
\leq \frac{1}{2} n^2
\]

As \( I(n) \geq 0, |I(n)| = I(n) \)

Putting \( c = \frac{1}{2} \), we obtain

\[
|I(n)| \leq cn^2, \quad \text{for } n \geq 1;
\]

That is, \( I(n) \) is \( O(n^2) \),

**Question 2.**

(i) 13.

(ii) 12.

(iii) 11.

(iv) \( 13 + 12 + 11 + 10 + \cdots + 1 = \frac{13 \times 14}{2} = 91 \).

(v) Yes! No, this is saying that every case is the worst case. In other sorts such as BUBBLE, MERGE and INSERTION we often do fewer than the number of comparisons required for the worst case.

(vi) Clearly

\[
S(n) = n - 1 + n - 2 + \cdots + 3 + 2 + 1 = \frac{n(n - 1)}{2}
\]

As \( S(n) = I(n) \), we see that \( S \) is \( O(n^2) \).

**Extra Question** What very simple modification can be made to insertion sort so that the worst case occurs when the given list is in alphabetical order? This is a starred question so no answer is provided.
Tutorial 4 Solutions

Question 1.

(i)

<table>
<thead>
<tr>
<th>List 5</th>
<th>List 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>butcherbird*</td>
<td>butcherbird*</td>
</tr>
<tr>
<td>cockatoo*</td>
<td>cockatoo*</td>
</tr>
<tr>
<td>crow*</td>
<td>crow*</td>
</tr>
<tr>
<td>currawong*</td>
<td>currawong*</td>
</tr>
<tr>
<td>eastern spinebill*</td>
<td>eastern spinebill*</td>
</tr>
<tr>
<td>fairy wren*</td>
<td>fairy wren*</td>
</tr>
<tr>
<td>GALAH</td>
<td>galah*</td>
</tr>
<tr>
<td>kookaburra</td>
<td>kookaburra*</td>
</tr>
<tr>
<td>magpie*</td>
<td>magpie*</td>
</tr>
<tr>
<td>rosella*</td>
<td>rosella*</td>
</tr>
<tr>
<td>wattlebird*</td>
<td>wattlebird*</td>
</tr>
<tr>
<td>wedge tailed eagle*</td>
<td>wedge tailed eagle*</td>
</tr>
</tbody>
</table>

(ii)

<table>
<thead>
<tr>
<th>LIST 1</th>
<th>LIST 2</th>
<th>LIST 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(n − 1 comparisons)</td>
<td>(n − 2 comparisons)</td>
</tr>
<tr>
<td>A₁</td>
<td>a₁*</td>
<td>a₂*</td>
</tr>
<tr>
<td>a₂</td>
<td>A₂</td>
<td>a₃</td>
</tr>
<tr>
<td>a₃</td>
<td>a₃</td>
<td>A₃</td>
</tr>
<tr>
<td>a₄</td>
<td>a₄</td>
<td>a₅</td>
</tr>
<tr>
<td>a₅</td>
<td>a₅</td>
<td>a₅</td>
</tr>
<tr>
<td>.</td>
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<td>.</td>
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<tr>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>aₙ</td>
<td>aₙ</td>
<td>aₙ</td>
</tr>
</tbody>
</table>

(iii) Total number of comparisons

\[
\begin{align*}
\text{Total number of comparisons} & = n - 1 + n - 2 + n - 3 + \cdots + 1 \\
& = \frac{n(n - 1)}{2}
\end{align*}
\]
(iv) $n!$

(v) The set has 3 elements and hence $3! = 6$ possible orderings.

(a) BANANA apple* pear banana* apple pear* 2 comparisons
(b) APPLE apple* apple* pear PEAR banana* banana pear* 3 comparisons
(c) Apple apple* apple* banana BANANA banana* pear pear* 3 comparisons
(d) PEAR BANANA apple* banana apple banana* apple pear* 3 comparisons
(e) PEAR APPLE apple* apple banana banana* pear pear* 3 comparisons
(f) BANANA apple* apple banana* pear pear* 2 comparisons

$$Q(3) = \frac{3 + 3 + 3 + 3 + 2 + 2}{6} = \frac{16}{6} = \frac{2}{3}$$

(vi) $Q(1) = 0$ and $1 \log 1 = 0$. Thus $Q(1) \leq \frac{3}{2} \times 1 \log 1$, as required.

When $n = 2$ there are 2 possible orderings and each required 1 comparison. So $Q(2) = \frac{1+1}{2} = 1$, while $2 \log 2 = 2$. So $Q(2) \leq \frac{3}{2} \times 2 \log 2$, as required.

When $n = 3$, $3 \log 3 > 3$ and so $\frac{3}{2} \times 3 \log 3 > 4 \rightarrow Q(3) = 2\frac{2}{3} \leq \frac{3}{2} \times 3 \log 3$, as required.

**Question 2.**

<table>
<thead>
<tr>
<th>JOAN</th>
<th>GARY</th>
<th>Chris*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gary</td>
<td>Chris</td>
<td>Gary*</td>
</tr>
<tr>
<td>Chris</td>
<td>Gwenda</td>
<td>Gwenda*</td>
</tr>
<tr>
<td>Sid</td>
<td>Joan*</td>
<td>Joan*</td>
</tr>
<tr>
<td>Norm</td>
<td>SID</td>
<td>Norm*</td>
</tr>
<tr>
<td>Gwenda</td>
<td>Norm</td>
<td>Sid*</td>
</tr>
</tbody>
</table>
Tutorial 5 Solutions

Question 1.
Graphs (b) and (d) have Eulerian paths, but are not Eulerian.
Graphs (c), (e) and (f) are Eulerian graphs.
Graph (a) does not contain an Eulerian path.

Question 2.
(i) For Graph (b), the ordered edge list is

\[ AB, AF, BC, BE, BF, CD, DE, EF. \]

As there are two vertices of odd degree, begin at vertex \( F \). The Eulerian path using Fleury’s algorithm is

\[ FA, AB, BC, CD, DE, EB, BF, FE. \]

For Graph (c) the ordered edge list is

\[ AE, AF, BC, BE, CD, DE, EF. \]

As every vertex has even degree, we start at Vertex \( A \). The Eulerian circuit is

\[ AE, EB, BC, CD, DE, EF, FA. \]

For Graph (f) the ordered edge list is

\[ AB, AF, BC, BD, BF, CD, DE, DF, EF. \]

As every vertex has even degree, we start at Vertex \( A \). The Eulerian circuit is

\[ AB, BC, CD, DB, BF, FD, DE, EF, FA. \]

Notice, that after reaching vertex \( F \) the first time, the edge \( FA \) is a bridge, therefore we choose \( FD \) as the next edge.

(ii) The ordered edge list is

\[ AB, AC, AD, AF, AG, AH, BC, BD, BF, BG, BH, \]
There are two vertices of odd degree, \(E\) and \(F\), so begin at vertex \(E\). The Eulerian path is

\[EF, FA, AB, BC, CA, AD, DB, BF, FC, CD, DF, FG, GA, AH, HB, BG, GC, CH, HD, DG, GH, HF.\]

**Question 3.**

(i) and (ii)

<table>
<thead>
<tr>
<th></th>
<th>Number of edges</th>
<th>Number of vertices</th>
<th>Number of vertices of odd degree</th>
<th>Number of vertices of even degree</th>
<th>(N_G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2.</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3.</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>4.</td>
<td>10</td>
<td>10</td>
<td>6</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>5.</td>
<td>9</td>
<td>10</td>
<td>10</td>
<td>0</td>
<td>18</td>
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(iii) Guess that \(N_G = 2E\). Each edge joins two vertices, and so it counts as 1 towards the degree of each of the two vertices. So when the degrees of all the vertices are added together, each edge has been counted twice.

(iv) The sum of the degrees \(N_G\) can be divided into the sum of the degrees of the vertices of even degree \(E_G\) plus the sum of the degrees of the vertices of odd degree \(O_G\). So

\[N_G = E_G + O_G \quad \text{or} \quad O_G = N_G - E_G.\]

Now, the sum of any number of even numbers is always even, so \(E_G\) is an even number, and \(N_G\) must be even as it is twice the numbers of edges. Therefore \(N_G - E_G\) must be even; i.e., \(O_G\) must be an even number. If there was an odd number of vertices of odd degree, then \(O_G\) would be an odd number, so there must be an even number of vertices of odd degree.

**Question 4.**
The least number of edges is zero - no vertex is joined to any other. The maximum number of edges would occur when every vertex is joined to every other vertex. This would be a complete graph with $V$ vertices, and so there would be $\frac{V(V-1)}{2}$ edges. (Could you prove this?)

If the graph is to be connected, then the minimum number of edges would be when the vertices are joined as in a chain. The number of edges would be $V - 1$. The maximum would be the same as above.
Tutorial 6 Solutions

Question 1.

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Question 2.

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Therefore a minimal spanning tree for G has weight 14.

Question 3.

(a) One possible edge list is:

\[ e_1 = \{v_1, v_2\} \]
\[ e_2 = \{v_2, v_8\} \]
\[ e_3 = \{v_3, v_4\} \]
\[ e_4 = \{v_4, v_5\} \]
\[ e_5 = \{v_7, v_8\} \]
\[ e_6 = \{v_1, v_3\} \]
\[ e_7 = \{v_6, v_7\} \]
\[ e_8 = \{v_7, v_9\} \]
\[ e_9 = \{v_2, v_4\} \]
\[ e_{10} = \{v_5, v_6\} \]
\[ e_{11} = \{v_5, v_{10}\} \]
\[ e_{12} = \{v_2, v_3\} \]
Therefore, a minimal spanning tree has weight 14.

(b) One possible edge list is:

\[ e_1 = \{E, F\} \quad e_2 = \{A, E\} \quad e_3 = \{B, F\} \quad e_4 = \{D, G\} \]
\[ e_5 = \{C, D\} \quad e_6 = \{C, G\} \quad e_7 = \{B, C\} \quad e_8 = \{A, B\} \]
\[ e_9 = \{B, E\} \quad e_{10} = \{C, F\} \quad e_{11} = \{D, H\} \quad e_{12} = \{G, H\} \]
\[ e_{13} = \{F, G\} \]

Therefore, a minimal spanning tree has weight 191.

(c) One possible edge list is:

\[ e_1 = \{1, 4\} \quad e_2 = \{1, 6\} \quad e_3 = \{3, 8\} \quad e_4 = \{5, 8\} \]
\[ e_5 = \{1, 2\} \quad e_6 = \{1, 8\} \quad e_7 = \{2, 7\} \quad e_8 = \{2, 3\} \]
\[ e_9 = \{7, 8\} \quad e_{10} = \{3, 4\} \quad e_{11} = \{6, 7\} \quad e_{12} = \{4, 5\} \]
\[ e_{13} = \{5, 6\} \quad e_{14} = \{2, 5\} \quad e_{15} = \{4, 7\} \quad e_{16} = \{3, 6\} \]

Therefore a minimal spanning tree for \( G \) has weight 61.

**Question 4.**

(i) If the edge (but not the vertices at its ends) is removed, the graph becomes disconnected; that is, the edge must be a bridge.
Suppose that an edge does not satisfy this condition. Then we could rub out the edge and get a connected graph for which we could construct a spanning tree. This tree would contain all the vertices of the original graph and so would be a spanning tree for $G$ which did not contain the given edge. Conversely, if the edge does satisfy the condition, then by the time the algorithm reaches that edge, the two ends will have different labels, and it would be added to the edge list for the tree.

(ii) The edge must be a loop.

If the edge were not a loop, then by listing it first we could apply the algorithm and make sure that the edge appeared in at least one spanning tree. Conversely, if the edge were a loop it could not be in any tree, since trees do not have loops.
Tutorial 7 Solutions

Question 1.

(i) \( xyz \) and \( x'yz; x'y'z' \) and \( x'y'z' \).

(ii) \( wx'yz \) and \( w'x'yz; wx'y'z; wxy'z \) and \( wx'y'z \).

(iii) \( w'x'y'z \) and \( wx'y'z; wx'y'z \) and \( wx'y'z \); \( wxy'z \) and \( wx'y'z \); \( wxy'z \) and \( wx'y'z \); \( wxy'y \) and \( wxy'y'z \).

(iv) \( wxy'y \) and \( wxy'y'z; wxy'y'z \) and \( wxy'y'z; wxy'y'z \) and \( wxy'y'z \); \( wxy'y \) and \( wxy'y'z \).

Question 2.

(i)
\[
xyz + x'y'z + x'y'z' + x'y'z \\
= (xyz + x'y)z + (x'y'z' + x'y'z') \\
= (x + x')yz + x'(y' + y)z' \\
= 1yz + x'1z' \\
= yz + x'z' \\
\]
using (f)

(ii)
\[
wx'y'z + wxy'z + w'x'y'z + wx'y'z + w'xyz' \\
= (wx'y'z + w'x'y)z + (wxy'z + w'x'yz) + w'xyz' \\
= (w + w')x'y'z + w(x + x')y'z + w'xyz' \\
= 1x'y'z + wly'z + w'xyz' \\
= x'y'z + wy'z + w'xyz'. \\
\]

(iii)
\[
w'x'y'z + w'x'y'z + wxy'z + wxy'z + wxy'y'z + wxyz \\
= (w'x'y'z + wxy'z) + (wxy'z + wxyz) + (wxy'y'z + wxy'y'z') \\
= (w' + w)x'y'z + wxy(z' + z) + wxy'(z + z') \\
= x'y'z + wxy + wxy' \\
= x'y'z + w(xy + y') \\
= x'y'z + wx. \\
\]

(iv)
\[
wxy'y'z + w'x'y'z + w'y'z + wy'z + w'xy'z + wxyz \\
= (wxy'y'z + wxyz) + (w'x'y'z + w'xy'z) + (w'y'z + wy'z) \\
= wx(y' + y)z + w'(x' + x)y'z + (w' + w)y'z \\
= wxz + w'y'z + y'z \\
= w(x + (w' + 1)y'z) \\
= wxz + y'z. \\
\]
Question 3.

(i) 

(ii) 

\[ x'y'z' + xy'z' + xyz' + xyz = x'y'z' + xy'z' + (xyz' + xyz) = x'y'z' + xy'z' + xy(z' + z) = x'y'z' + xy'z' + xy \]

There are other solutions.

(iii) 

Question 4.

(i) 

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(ii) $w'xyz + wx'y'z + wx'yz + wxy'z' + wxy'z + wxyz' + wxyz$
(iii)

(iv)

\[ w'xyz + wx'y'z + wx'yz' + wxy'z' + wxy'z + wxyz' + wxyz = (w'xyz + wxyz) + (wx'y'z + wx'yz) + (wx'yz' + wxyz') + (wxy'z' + wxy') \]

\[ = (w' + w)xyz + wx'(y' + y)z + w(x' + x)yz' + wxy'(z' + z) \]

\[ = xyz + wx'z + wyz' + wxy' \]

(v)
Tutorial 8 Solutions

Question 1.

(a) The Boolean expression is \( x'y' + xy \) which is already a minimal representation. The gate implementation is as follows:

(b) The Boolean expression is \( x'y'z' + x'y'z + xyz' + xyz \) which has a minimal representation as \( x'z' + yz' \). The gate implementation is as follows:

(c) The Boolean expression is \( x'yz + xy'z + xyz' + xyz \), which has minimal representation \( xy + xz + yz \). The gate implementation is:
**Question 2.**

(a)

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(b) The Boolean expression for the output $c$ is $xy$, and the gate implementation is:

```
 \[ x \quad y \quad xy = c \]
```

The Boolean expression for the output $s$ is $x'\,y + xy'$ and the gate implementation is:

```
   x
   y
   \hline
   x' \quad y' \quad xy' \quad xy' = s
```

(c)
Question 3.

(a) 

<table>
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<th>c</th>
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(b) 

\[ z \]
\[ x \]
\[ y \]
\[ \text{HA} \]
\[ s \]
\[ c \]

\[ S \]
\[ C \]

Question 4.

(a) 

\[ x \]
\[ y \]
\[ xy \]
\[ (xy)' \]

(b) 

\[ x \]
\[ y \]
\[ x+y \]
\[ (x+y)' \]

Question 5.

(a) 

\[ x \]
\[ z \]
\[ y \]
\[ (xz)' \]
\[ xz \]
\[ (xzy')'=y+(xz)' \]

(b) 

\[ x \]
\[ z \]
\[ y \]
\[ x' \]
\[ z' \]
\[ (x'+z')'=xz \]
\[ (xz)' \]
\[ (y+(xz)')'=y+(xz)' \]
Tutorial 9 Solutions

Question 1.

Therefore the minimal representation is

\[ w'xyz + wyz' + w'xyz' + w'zy' + x'z' \]

Question 2.

Therefore the minimal representation is

\[ w'xy'z' + xyz + x'y'z + x'y'z + wy + wx' \]

Question 3.

(a)

From (A) we obtain the minimal representation \( E = w'xz + w'xy' + wy'z \)
and from (B) we get \( F = w'xz + xy'z + wx'y' \)

(b) Both expressions, \( E \) and \( F \), are minimal as both are the sum of three product terms, and both contain nine literals.
Question 4.
(a) and (b)

(c)(i)

A minimal representation is $vwxy + wxyz' + w'x'z' + v'y'z + x'y'$

(ii)

A minimal representation is $w'x'y'z' + x'yz + wxz' + vw'x'$
Practice Class 1 Solutions

Question 1.

a) For any \( x \in A - B \), we have from its definition \( x \in A \) and \( x \notin B \). Hence \( x \in A \) and \( x \in B' \), implying \( x \in A \cap B' \). In other words we have \( A - B \subseteq A \cap B' \). Conversely for any \( x \in A \cap B' \) we have \( x \in A \) and \( x \in B' \) and thus \( x \in A - B \). This means \( A \cap B' \subseteq A - B \). Combined with the previous result \( A - B \subseteq A \cap B' \) we conclude \( A - B = A \cap B' \).

b) From (a) we see that

\[
(A \cup B) - (C - A) = (A \cup B) - (C \cap A') \quad \text{via (a)}
\]

\[
= (A \cup B) \cap (C \cap A')' \quad \text{via (a)}
\]

\[
= (A \cup B) \cap (C' \cup A'') \quad \text{de Morgan’s laws}
\]

\[
= (A \cup B) \cap (C' \cup A) \quad \text{double complement identity}
\]

c) From (b) we see that

\[
(A \cup B) - (C - A) = (A \cup B) \cap (C' \cup A) \quad \text{via (b)}
\]

\[
= A \cup (B \cap C') \quad \text{distributivity}
\]

\[
= A \cup (B - C) \quad \text{via (a)}
\]

Question 2.

a) \( \mathcal{P}(A) = \{ \emptyset, \{1\}, \{2\}, \{1, 2\} \} \).

(b) \( A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\} \).

c) \( \mathcal{P}(A) \times A = \{(\emptyset, 1), (\emptyset, 2), ([1], 1), ([1], 2), ([2], 1), ([2], 2), ([1, 2], 1), ([1, 2], 2)\} \).

Question 3.

(a) \( \sum_{i=1}^{10} i \)

(b) \( \sum_{k=1}^{n} (2k - 1) \)

c) \( \sum_{m=1}^{k} m^3 \)

(d) \( \sum_{j=2}^{k/2} (2j)^2 \)

Question 4.

(a) \( 3^2 + 4^2 + 5^2 + 6^2 \)

(b) \( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \)

c) \( 1 + 3 + 3^2 + 3^3 \)

(d) \( 1 + \frac{3}{2} + 2 + \frac{5}{2} \)
Question 5.

Let $P_n$ be the statement $1 + 3 + 9 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}$ for each $n \in \mathbb{N}$. So $P_n$ is $\sum_{m=1}^{n} 3^{m-1} = \frac{3^n - 1}{2}$.

When $n = 1$, $\sum_{m=1}^{1} 3^{m-1} = 3^0 = 1$, and $\frac{3^1 - 1}{2} = 1$ also. So $P_1$ is true. Assume $\sum_{m=1}^{k} 3^{m-1} = \frac{3^k - 1}{2}$ for some $k \in \mathbb{N}$. (We must show $\sum_{m=1}^{k+1} 3^{m-1} = \frac{3^{k+1} - 1}{2}$.)

\[
\begin{align*}
\sum_{m=1}^{k+1} 3^{m-1} &= [1 + 3 + 9 + \cdots + 3^{k-1}] + 3^{(k+1)-1} \\
&= \sum_{m=1}^{k} 3^{m-1} + 3^k \\
&= \frac{3^k - 1}{2} + 3^k, \text{ using assumption} \\
&= \frac{3^k - 1 + 2 \cdot 3^k}{2} \\
&= \frac{3^{k+1} - 1}{2}, \text{ as required.}
\end{align*}
\]

Therefore, by the Principle of Mathematical Induction, $1 + 3 + 9 + \cdots + 3^{n-1} = \frac{3^n - 1}{2}$ for each $n \in \mathbb{N}$.

Question 6.

Let $S_n$ be the statement $\sum_{i=1}^{n} i^3 = \left[ \frac{n(n + 1)}{2} \right]^2$, for all integers $n \geq 1$.

When $n = 1$, $\sum_{i=1}^{1} i^3 = 1^3 = 1$ and $\left[ \frac{1(1 + 1)}{2} \right]^2 = 1^2 = 1$ also. So $S_1$ is true.

Assume $\sum_{i=1}^{k} i^3 = \left[ \frac{k(k + 1)}{2} \right]^2$ for some integer $k \geq 1$.

Now to show $\sum_{i=1}^{k+1} i^3 = \left[ \frac{(k + 1)(k + 2)}{2} \right]^2$. 
\[
\sum_{i=1}^{k+1} i^3 = \left[ 1^3 + 2^3 + 3^3 + \cdots + k^3 \right] + (k+1)^3
\]
\[
= \sum_{i=1}^{k} i^3 + (k+1)^3
\]
\[
= \left[ \frac{k(k+1)}{2} \right]^2 + (k+1)^3, \quad \text{by assumption}
\]
\[
= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}
\]
\[
= \frac{(k+1)^2(k^2 + 4(k+1))}{4}
\]
\[
= \frac{(k+1)^2(k+2)^2}{4}, \quad \text{as required.}
\]

Therefore, by the Principle of Mathematical Induction, \( \sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2 \) for \( n \geq 1 \).

**Question 7.**

Let \( S_n \) be the statement \( \sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1} \), for all integers \( n \geq 1 \).

When \( n = 1 \), \( \sum_{j=1}^{1} \frac{1}{j(j+1)} = \frac{1}{1(1+1)} = \frac{1}{2} \) and \( \frac{1}{1+1} = \frac{1}{2} \) also. So \( S_1 \) is true.

Assume \( \sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{k}{k+1} \) for some integer \( k \geq 1 \).

Now to show \( \sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \frac{k+1}{k+2} \).

\[
\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \left[ \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{k(k+1)} \right] + \frac{1}{(k+1)(k+2)}
\]
\[
= \sum_{j=1}^{k} \frac{1}{j(j+1)} + \frac{1}{(k+1)(k+2)}
\]
\[
= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}, \quad \text{by assumption}
\]
\[
= \frac{k(k+2) + 1}{(k+1)(k+2)}
\]
\[
= \frac{(k+1)^2}{(k+1)(k+2)}
\]
\[
= \frac{k + 1}{k+2}, \quad \text{as required.} \]
Therefore, by the Principle of Mathematical Induction, \( \sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1} \) for \( n \geq 1 \).

**Question 8.**
Let \( S_n \) be the statement \( 2n + 1 < 2^n \), for all integers \( n \geq 3 \).
When \( n = 3, 2n + 1 = 7 \) and \( 2^n = 8 > 7 \). So \( S_3 \) is true.
Assume \( 2k + 1 < 2^k \) for some integer \( k \geq 3 \).
Now to show \( 2k + 3 < 2^{k+1} \).

\[
2k + 3 = 2k + 1 + 2 < 2^k + 2^k, \quad \text{by assumption}
\]
\[
< 2^k + 2^k \quad \text{since } k > 1
\]
\[
= 2 \times 2^k
\]
\[
= 2^{k+1}, \quad \text{as required.}
\]
Therefore, by the Principle of Mathematical Induction, \( 2n + 1 < 2^n \), for all integers \( n \geq 3 \).

**Question 9.**
Let \( S_n \) be the statement \( n^2 < 2^n \), for all integers \( n \geq 5 \).
When \( n = 5, n^2 = 25 \) and \( 2^n = 32 > 25 \). So \( S_5 \) is true.
Assume \( k^2 < 2^k \) for some \( k \geq 5 \).
Now to show \((k + 1)^2 < 2^{k+1}\).

\[
(k + 1)^2 = k^2 + 2k + 1 < 2^k + 2k + 1, \quad \text{by assumption}
\]
\[
< 2^k + 2^k, \quad \text{by Question 8}
\]
\[
= 2 \times 2^k
\]
\[
= 2^{k+1}, \quad \text{as required.}
\]
Therefore, by the Principle of Mathematical Induction, \( n^2 < n^n \), for all integers \( n \geq 5 \).

**Question 10.**
Let \( S_n \) be the statement \( 4^n - 1 \) is divisible by 3, for integers \( n \geq 1 \).
When \( n = 1, 4^n - 1 = 4 - 1 = 3 \) which is divisible by 3. So \( S_1 \) is true.
Assume \( 4^k - 1 \) is divisible by 3 for some integer \( k \geq 1 \). That is, \( 4^k - 1 = 3m \) for some integer \( m \).
To show \( 4^{k+1} - 1 \) is divisible by 3.

\[
4^{k+1} - 1 = 4 \times 4^k - 1 = 4 \times (3m + 1) - 1, \quad \text{by assumption}
\]
\[
= 12m + 3
\]
\[
= 3(4m + 1) \quad \text{where } 4m + 1 \text{ is an integer.}
\]
So $4^{k+1} - 1$ is divisible by 3.

Therefore, by the Principle of Mathematical Induction, $4^n - 1$ is divisible by 3, for all integers $n \geq 1$.

**Question 11.**

Let $S_n$ be the statement $n^3 - n$ is divisible by 6, for all integers $n \geq 1$.

When $n = 1$, $n^3 - n = 1^3 - 1 = 0$ which is divisible by 6. So $S_1$ is true.

Assume $k^3 - k$ is divisible by 6, for some integer $k \geq 1$. That is, $k^3 - k = 6m$ for some integer $m$.

To show $(k + 1)^3 - (k + 1)$ is divisible by 6.

\[
(k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1)
= k^3 + 3k^2 + 2k
= (6m + k) + 3k^2 + 2k \text{ by assumption}
= 6m + 3k^2 + 3k
= 6m + 3k(k + 1)
= 6m + 6p \text{ for some integer } p \text{ because either } k \text{ or } k + 1 \text{ is even}
= 6(m + p) \text{ where } m + p \text{ is an integer.}
\]

So $(k + 1)^3 - (k + 1)$ is divisible by 6.

Therefore, by the Principle of Mathematical Induction, $n^3 - n$ is divisible by 6, for all integers $n \geq 1$.

**Question 12.**

Let $S_n$ be the statement $3^n + 7^n - 2$ is divisible by 8, for $n \in \mathbb{N}$.

Verify $S_1 : 3^1 + 7^1 - 2 = 8$ which is clearly divisible by 8. $S_1$ is true.

Assume $S_k$, i.e. $3^k + 7^k - 2 = 8M$ where $M$ is some natural number (i.e. $M \in \mathbb{N}$).

Prove $S_{k+1}$: we must show that $3^{k+1} + 7^{k+1} - 2$ is divisible by 8.

\[
3^{k+1} + 7^{k+1} - 2 = 3 \times 3^k + 7 \times 7^k - 2
= 3(3^k + 7^k) + 4 \times 7^k - 2
= 3(2 + 8M) + 4 \times 7^k - 2, \text{ by } S_k
= 8 \times 3M + 4(7^k + 1)
\]

Now $7^k$ is odd so $7^k + 1$ is even and we can write $7^k + 1 = 2p$ where $p \in \mathbb{N}$.

Therefore

\[
3^{k+1} + 7^{k+1} - 2 = 8 \times 3 \times M + 4 \times 2 \times p
= 8(3M + p)
\]

and so $3^{k+1} + 7^{k+1} - 2$ is divisible by 8, i.e. $S_{k+1}$ is true. Therefore $S_n$ is true for all $n \in \mathbb{N}$.
Practice Class 2 Solutions

Question 1.
(i) \[ f(x) = (4x^3 - 3x^2 + x) - 2 \]
\[ = x(4x^2 - 3x + 1) - 2 \]
\[ = x(x(4x - 3) + 1) - 2 \]

(ii) \[ f(1) = 1(1(4 \times 1 - 3) + 1) - 2 \]
\[ = 1(1 \times 2) - 2 \]
\[ = 2 - 2 \]
\[ = 0 \]

\[ f(-1) = -1(-1(4 \times -1 - 3) + 1) - 2 \]
\[ = -1(-1 \times -7 + 1) - 2 \]
\[ = -8 - 2 \]
\[ = -10 \]

Question 2.
(i) \[ f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \]
\[ = a_0 + x(a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4) \]
\[ = a_0 + x(a_1 + x(a_2 + a_3x + a_4x^2 + a_5x^3)) \]
\[ = a_0 + x(a_1 + x(a_2 + x(a_3 + a_4x + a_5x^2))) \]
\[ = a_0 + x(a_1 + x(a_2 + x(a_3 + x(a_4 + a_5x)))) \]

(ii) 5

(iii) \[ 5 + 4 + 3 + 2 + 1 = 15. \]

(iv) 5

(v) 5

(vi) Yes. Although the same number of additions/subtractions are used in each method, there are fewer multiplications used in Horner’s method.
Question 3.

\[ |f(x)| = |2x^3 + 3x^2 + x - 1| \]
\[ \leq 2|x^3| + 3|x^2| + |x| + |1|, \text{ by triangle inequality} \]
\[ = 2x^3 + 3x^2 + x + 1, \quad \text{if } x \geq 0 \]
\[ \leq 2x^3 + 3x^3 + x^3 + x^3, \quad \text{if } x \geq 1 \]
\[ = 7x^3 \]

Therefore \( f(x) = O(x^3) \).
Question 4.

(i) We know \( n \leq 2^n \), for \( n \geq 1 \). Taking logarithms to base 2 of both sides gives

\[
\log n \leq \log(2^n) = n \log 2 = n, \quad \text{for all } n \geq 1.
\]

(ii) \( \log n = O(n) \) if there exist constant \( M \) and positive constant \( C \) such that

\[
|\log n| \leq C \cdot |n|, \quad \text{for all } n \geq M.
\]

(iii) If we choose \( C = 1 \) and \( M = 1 \), then from (i) the condition specified in (ii) is satisfied. Hence \( \log n = O(n) \).

(iv) Since \( \log n \geq 1 \) for all \( n \geq 2 \), we see that \( |1| \leq |\log n| \), for all \( n \geq 2 \).

By choosing \( C = 1 \) and \( M = 2 \) we conclude from the definition of \( O \) that \( 1 = O(\log n) \).

(v) Multiplying the inequality in (iv) by \( n \), we easily arrive at

\[
|n| \leq |n \log(n)|, \quad \text{for all } n \geq 2.
\]

By again choosing \( C = 1 \) and \( M = 2 \), as in (iv), we conclude that \( n = O(n \log(n)) \).

(vi) From the inequality in (i) we have immediately \( |n \cdot \log(n)| \leq |n \cdot n| \), for \( n \geq 1 \), implying immediately \( n \log(n) = O(n^2) \).

(vii)

\[
n! = n(n-1)(n-2)(n-3)\ldots5.4.3.2.1
\]

\[
= (n(n-1)(n-2)\ldots5).4.3.2.1, \quad \text{for } n \geq 4
\]

\[
> \underbrace{2.2.2\ldots2}_{n-4 \text{ terms}} \times 24
\]

\[
= 2^{n-4} \times 24
\]

\[
> 2^{n-4} \times 2^4, \quad \text{as } 2^4 = 16 < 24
\]

\[
= 2^n
\]

So \( n! > 2^n, \quad n \geq 4 \).

(viii) If we choose \( C = 1 \) and \( M = 4 \), then (vii) implies

\[
|2^n| \leq C \cdot |n|!, \quad \text{for all } n \geq M.
\]

Hence \( 2^n = O(n!) \).
**Question 5.**

\[ |f(n)| = |n^3 - 4n^2 - n + 1| \]

\[ \leq |n^3| + 4|n^2| + |n| + |1|, \text{ by triangle inequality} \]

\[ = n^3 + 4n^2 + n + 1, \quad \text{since } n \geq 0 \]

\[ \leq n^3 + 4n^3 + n^3 + n^3, \quad \text{if } n \geq 1 \]

\[ = 7n^3 \]

\[ |f(n)| = |n^3 - 4n^2 - n + 1| \]

\[ = |(n^3 + 1) - (4n^2 + n)| \]

\[ \geq |n^3 + 1| - |4n^2 + n|, \text{ by triangle inequality} \]

\[ \geq n^3 + 1 - 4n^2 - n, \quad n \geq 0 \]

\[ \geq n^3 + 1 - 5n^2, \quad \text{if } n \geq 1 \]

\[ \geq n^3 - 5n^2 \]

\[ \geq \frac{1}{2}n^3, \quad \text{if } n \geq 10 \]

Therefore \( f(n) = \theta(n) \).

**Question 6.**

\[ |f(n)| = \left| \frac{4n^2 + 2 \log_2 n}{n} \right| \]

\[ = \frac{4n^2 + 2 \log_2 n}{n}, \quad \text{since each term is } \geq 0 \]

\[ \leq \frac{4n^2 + 2n}{n}, \quad \text{since } \log_2 n \leq n \]

\[ \leq \frac{4n^2 + 2n^2}{n}, \quad \text{since } n \geq 1 \]

\[ = \frac{6n^2}{n} \]

\[ = 6n \]

\[ |f(n)| = \frac{4n^2 + 2 \log_2 n}{n}, \quad \text{since each term is } \geq 0 \]

\[ \geq \frac{4n^2}{n}, \quad \text{since } \log_2 n \geq 0 \quad \text{for } n \geq 1 \]

\[ = 4n \]

Therefore \( f(n) = \theta(n) \).
Question 7.

\[ |f(n)| \leq n^3 + 2n^2 + \log n + 3 \]
\[ \leq n^3 + 2n^3 + n^3 + 3n^3, \text{ if } n \geq 1 \]
\[ = 7n^3. \]

So \(|f(n)| \leq 7n^3 \text{ if } n \leq 1\).

\[ |f(n)| \geq f(n) \]
\[ \geq n^3 - 2n^2 - 3 \text{ as } n \geq 1 \]
\[ \geq 0 \text{ for } n \geq 3. \]

We want \(2n^2\) and 3 to be less than \(\frac{n^3}{4}\). For \(2n^2 \leq \frac{n^3}{4}\) we must have \(n \geq 8\), and for \(3 \leq \frac{n^3}{4}\) we must have \(n \geq 3\).

So \(|f(n)| \geq n^3 - \frac{n^3}{4} - \frac{n^3}{4} = \frac{n^3}{2} \text{ if } n \geq 8\).

Combining (1) and (2) we see that if \(n \geq 8\) then

\[ \frac{1}{2}n^3 \leq |f(n)| \leq 7n^3. \]

So choose \(d = \frac{1}{2}, \ c = 7 \text{ and } M = 8. \) Then \(f(n) = \Theta(n^3).\)
Practice Class 3 Solutions

Question 1.
(i) $\sim p \land q$  (ii) $p \land \sim q$  (iii) $\sim p \lor (p \land q)$

Question 2.
(i) $T$  (ii) $T$  (iii) $T$  (iv) $T$  (v) $T$

Question 3.
(i)

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(ii) (a) $\sim (p \land q) \lor r$  (b) $\sim p \lor (p \land q)$  $\equiv \sim p \lor q$

Question 4.
(i)

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(ii) To show $p \rightarrow q$ is false, we must show $p$ is true and $q$ is false.

Question 5.
(i)

$\sim (p \land q) \equiv \sim p \lor \sim q$

$\equiv \sim p \lor q$

(ii)

$\sim (\sim p \lor q) \equiv \sim \sim p \land \sim q$

$\equiv p \land \sim q$

(iii)

$\sim (\sim p \land \sim q) \equiv \sim \sim p \lor \sim q$

$\equiv p \lor q$
$(iv)$

\[\sim ((p \land q) \lor \sim q) \equiv \sim (p \land q) \land \sim q\]

\[\equiv (\sim p \lor \sim q) \land q\]

\[\equiv (\sim p \land q) \lor (\sim q \land q)\]

\[\equiv \sim p \land q\]
Practice Class 4 Solutions

Question 1.

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Rows 3, 5 and 7 are the critical rows. In these rows r has T’s so the argument is valid.

Question 2.

1. \( p \) premise
   \( s \rightarrow \sim p \) premise
   \( \therefore \sim s \) modus tollens

2. \( t \lor s \) premise
   \( \sim s \) from 1.
   \( \therefore t \) disjunctive syllogism

3. \( \sim s \rightarrow u \land q \) premise
   \( \sim s \) from 1.
   \( \therefore u \land q \) modus ponens

4. \( u \land q \)
   \( \therefore q \) conjunctive simplification

5. \( t \) from 2
   \( q \) from 4
   \( \therefore t \land q \) conjunctive addition

6. \( t \land q \rightarrow r \) premise
   \( t \land q \) from 5.
   \( \therefore r \) modus ponens
Question 3.

1. \( p \to t \)  given premise
   \( \sim t \)  given premise
   \( \therefore \sim p \)  by modus tollens

2. \( \sim p \)  from 1.
   \( \therefore \sim p \lor q \)  by disjunctive addition

3. \( \sim p \lor q \to r \)  given premise
   \( \sim p \lor q \)  from 2.
   \( \therefore r \)  by modus ponens

4. \( \sim p \)  from 1.
   \( r \)  from 3.
   \( \therefore \sim p \land r \)  by conjunctive addition

5. \( \sim p \land r \to \sim s \)  given premise
   \( \sim p \land r \)  from 4.
   \( \therefore \sim s \)  by modus ponens

6. \( s \lor \sim q \)  given premise
   \( \sim s \)  from 5.
   \( \therefore \sim q \)  by disjunctive syllogism

Question 4.

1. The last column in the truth table below

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( Q \to P )</th>
<th>( P \to (Q \to P) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( F )</td>
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</tr>
<tr>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

shows that \( P \to (Q \to P) \) is a tautology.

2. The last column of the truth table below

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \to Q )</th>
<th>( \sim P )</th>
<th>( \sim Q )</th>
<th>( \sim Q \to \sim P )</th>
<th>( \sim Q \to \sim P \to (P \to Q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
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<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

is constructed from the 3rd and 6th columns, and it shows that \( \sim Q \to \sim P \to (P \to Q) \)
is always true, hence it is a tautology.
3. We need to show that \((\forall x, P(x)) \rightarrow (\forall x, Q(x))\) is true when \((\forall x, P(x) \rightarrow Q(x))\) is true. So we just need to show \((\forall x, Q(x))\) is true if \((\forall x, P(x))\) is true and \((\forall x, P(x) \rightarrow Q(x))\) is also true.

If \((\forall x, P(x))\) is true, then \(P(x)\) is true for all \(x\). If furthermore \((\forall x, P(x) \rightarrow Q(x))\) is also true, then, for any \(x\), \(P(x)\) true will imply \(Q(x)\) is true. But we have just shown \(P(x)\) is always true, then so is \(Q(x)\). That is \((\forall x, Q(x))\) is true. From the argument in the previous paragraph we conclude that \((\forall x, P(x) \rightarrow Q(x)) \rightarrow (\forall x, P(x)) \rightarrow (\forall x, Q(x))\) is always true and is thus a tautology.

**Question 5.**

(i) Valid.

First premise “all people are mice” means that the set of people is a subset of mice. Second premise “all mice are mortal” means that the set of mice is a subset of the set of mortals, see the diagram below.

Hence the set of people is a subset of the set of mortals, implying the conclusion “all people are mortal” is valid.

(ii) Valid.

First premise “all teachers occasionally make mistakes” means the set of teachers is a subset of the set of beings who occasionally make mistakes. Second premise “no gods ever make mistakes” implies that the set of gods is disjoint from the set of beings who occasionally make mistakes, and is thus also disjoint from the set of teachers, see the diagram below.
Hence the conclusion “no teachers are gods” is valid.

**Question 6.**

The order of premises should be 4, 1, 5, 3, 6, 2.

4. All the animals in the yard are mine.

1. I trust every animal that belongs to me. So (from 4.) I trust all the animals in the yard.

5. I admit every animal that I trust into my study. So (from the previous conclusion) I admit all the animals in the yard into my study.

3. I admit no animals into my study unless they will beg when told to do so. So (from the previous conclusion) all animals in the yard will beg when told to do so.

6. The only animals that are really willing to beg when told to do so are dogs. So (from the previous conclusion) all animals in the yard are dogs.

2. Dogs gnaw bones. So (from the previous conclusion) all animals in the yard gnaw bones.
Practice Class 5 Solutions

Question 1.

(i) 1. \( f(1) = F, \ f(2) = E, \ f(3) = D, \ f(4) = C, \ f(5) = B, \ f(6) = A; \)
2. \( f(1) = F, \ f(2) = E, \ f(3) = A, \ f(4) = B, \ f(5) = C, \ f(6) = D. \)

(ii) 1. \( f(1) = B, \ f(2) = E, \ f(3) = C, \ f(4) = D, \ f(5) = A; \)
2. \( f(1) = B, \ f(2) = C, \ f(3) = E, \ f(4) = D, \ f(5) = A. \)

Question 2.

(i) There are the same number of edges joining vertex \( i \) to vertex \( j \) as there are edges joining vertex \( j \) to vertex \( j \).

(ii) There are no loops in the graph.

(iii) \[
\begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Question 3.

\( K_n \) is planar for \( n \leq 4 \):

Using Kuratowski’s Theorem, \( K_5 \) is non–planar. But for each \( n > 5 \), \( K_n \) is also non–planar, as it is obtained by adding \( n – 5 \) vertices to \( K_5 \) and then adding extra edges to join up all pairs of vertices.

Question 4.

These are not isomorphic, since, in example (i) all vertices have degree 1 or 2, in example (ii) there is a vertex of degree 3 and in example (iii) there is vertex of degree 4.
Question 5.

By Kuratowski’s Theorem, the graph must be obtained by adding some (or no) edges to $K_{3,3}$ or must be obtained by adding one extra vertex to $K_5$. In the first situation, we may keep adding edges, until adding an edge would create multiple edges (14 non–isomorphic possibilities). In the latter case, the extra vertex can either divide an existing edge, or be connected to 1, 2, 3, 4 or 5 vertices (6 possibilities).
Practice Class 6 Solutions

Question 1.
For $n = 2$

For $n = 3$

Question 2.
(a)  (i)

(b) 

(ii)
Question 3.

(b) (i) $D, B, E, A, F, C, G$.
(ii) $A, C, E, D, B$.

(c) For (a) the traversal list is 9,17,20,25. For (b) it is 9,13,17,20,25,30,35.
(ii) The average number of comparisons is $(3 + 3 + 2 + 2 + 3 + 3)/6 = 8/3$.
(d) (i) 21 comparisons (ii) 12 comparisons (iii) 10 comparisons

(e) (i) The most number of comparisons would occur if the numbers were in increasing or decreasing order.
(ii) The following order would give the least number of comparisons (when each level of the binary tree is full): 8, 4, 12, 2, 6, 10, 14, 1, 3, 5, 7, 9, 11, 13, 15.
Practice Class 7 Solutions

Question 1.

(a) \[115 = 3 \times 38 + 1\]
\[38 = 3 \times 12 + 2\]
\[12 = 3 \times 4 + 0\]
\[4 = 3 \times 1 + 1\]
\[1 = 3 \times 0 + 1\]

So \(115 = 11021_3\).

(b) \(115 = 430_5\)

(c) \(206 = 1311_5\)

(d) \(12 = 110_3\)

(e) \(67 = 1000011_2\)

(f) \(67 = 1003_4\)

(g) The easiest algorithm for converting a fraction in base 10 to a fraction in base \(b\) is to multiply the number by the base, \(b\). The integer part of the result is the first digit of the answer, and the fractional part of the result is then multiplied by the base \(b\) again. This procedure is continued until the fractional part is zero, or repeats itself. For example, 0.12 to base 5:

\[0.12 \times 5 = 0.6\]

So ‘0’ is the first digit of the base 5 representation, and the fractional part is 0.6, so continue...

\[0.6 \times 5 = 3.0\]

So ‘3’ is the second digit, and the fractional part is now ‘0’, so stop. The answer is

\[0.12 = 0.03_5\]

(h) \[0.6 \times 3 = 1.8, \text{ so 1 is the first digit}\]
\[0.8 \times 3 = 2.4, \text{ so 2 is the second digit}\]
\[0.4 \times 3 = 1.2, \text{ etc}\]
\[0.2 \times 3 = 0.6 \ldots \text{ at this stage, we are going to repeat the last 4 digits.}\]

So \(0.6 = 0.1210_3\).
(i)  

\[
\begin{align*}
0.56 \times 4 &= 2.24 \\
0.24 \times 4 &= 0.96 \\
0.96 \times 4 &= 3.84 \\
0.84 \times 4 &= 3.36 \\
0.36 \times 4 &= 1.44 \\
0.44 \times 4 &= 1.76 \\
0.76 \times 4 &= 3.04 \\
0.04 \times 4 &= 0.16 \\
0.16 \times 4 &= 0.64 \\
0.64 \times 4 &= 2.56 \leftarrow \text{repeat...}
\end{align*}
\]

So \(0.56 = \overline{0.2033113002}_4\)

**Question 2.**

(a) \(11100_2\)  
(b) \(10001_2\)  
(c) \(111100_2\)  
(d) \(10110101_2\)  
(e) \(111101010_2\)  
(f) \(10111101010_2\)

**Question 3.**

(a) \(110001_2\)  
(b) \(10100_2\)  
(c) \(10011000_2\)  
(d) \(1000000_2\)  
(e) \(110001_2\)  
(f) \(101010_2\)

**Question 4.**

(a) \(110_2\)  
(b) \(1001_2\)  
(c) \(1011110_2\)  
(d) \(100000_2\)  
(e) \(10_2\)  
(f) \(110100_2\)

**Question 5.**

(a)

\[
\begin{array}{c}
10111 \\
1111 \times \\
----- \\
10111 \\
10111 \\
10111 \\
10111 \\
----- \\
101011001
\end{array}
\]

(b)

\[
\begin{array}{c}
10100 \\
111 \times \\
----- \\
10100 \\
10100 \\
10100 \\
10100 \\
----- \\
10001100
\end{array}
\]
Answer: 1001₂

Answer: 1100₂
Question 1.

(i) The relation $R$ is obviously given by

$$R = \{ (x, y) : x, y \in \mathbb{R}, |x - y| \leq 1 \}$$

The area $R$ in the plane $\mathbb{R} \times \mathbb{R}$ is between the line $y - x = 1$ and the the line $y - x = -1$, and is depicted in the following diagram

(ii) Yes. Because $xRx$ due to $|x - x| \leq 1$ for any $x$.

(iii) Yes. Because

$$xRy \iff |x - y| \leq 1 \iff |y - x| \leq 1 \iff yRx$$

(iv) No. Because $R$ is not transitive. If for instance we choose $x = -1$, $y = 0$ and $z = 1$, then we have $xRy$ and $yRz$ because $|x - y| \leq 1$ and $|y - z| \leq 1$. However it is clear that $xRz$ because $|x - z| = |(-1) - 1| = 2 \not\leq 1$. $R$ is not transitive means that $R$ can not be an equivalence relation.

(v) No. Because $R$ is not anti-symmetric. If for instance we choose $x = 1$ and $y = 0$, then we have $xRy$ and $yRx$ but $x$ is not equal to $y$. Relation $R$ is not anti-symmetric means that $R$ can not be a partial order relation.

Question 2.

(i) (a) $R$ is reflexive because, for any $x \in A$, $x$ divides itself ($x| x$), implying $xRx$. 

![](image-url)
(b) $R$ is antisymmetric if for any $x, y \in A$, if $xRy$ and $yRx$, i.e., $x|y$ and $y|x$. From the definition of $R$, there exist integers $a$ and $b$ such that $y = xa$ and $x = yb$, hence $y = xa = (yb)a = y(ab)$, that is $ab = 1$. Since $a, b \in \mathbb{N}$, $a = b = 1$, $y = x$.

(c) $R$ is transitive: for any $x, y, z \in A$, if $xRy$ and $yRz$, i.e. $x|y$ and $y|z$ from the definition of $R$, then there exist integers $a$ and $b$ such that $y = xa$ and $z = yb$. Hence $z = x(ab)$ with integer $ab$, implying $x|z$ or $xRz$. This proves that $R$ is transitive.

From (a), (b) and (c) above we conclude that $R$ defined by $(\ast)$ is a partial order relation.

(ii)

$$R = \{(1, 1), (2, 2), (3, 3), (9, 9), (18, 18), (1, 2), (1, 3), (1, 9), (1, 18), (2, 18), (3, 9), (3, 18), (9, 18)\}$$

![Diagram]

**Question 3.**

(i) We look at the number 5 first. Since no numbers in $A$ can divide 5 because 5 is a prime number, i.e. $x \preceq 5$ does not hold for any $x \in A$, (no $x$ precedes 5), we conclude 5 is a minimal element of $A$.

Likewise, both 2 and 3 are minimal elements of $A$.

(ii) Since $2|4$ means $2 \preceq 4$, we see immediately that 4 is not a minimal element of $A$.

(iii) For any $x \in A$, since $2x \in A$ and $x|(2x)$, we have $x \preceq 2x$, i.e. $x$ precedes $2x$ under $\preceq$. Hence $x$ can not be a maximal element. Since $x$ is arbitrary element in $A$, we conclude the $A$ does not have any maximal elements.

(iv) The fact that $2, 3 \in A$ and 2 is not comparable with 3 means 2 is not a least element of $A$. 
(v) If $p \geq 2$ is a prime number, then $p \in A$ and no numbers in $A$ can ever divide $p$. In other words, we can not find a number $x \in A$ such that $x|p$, i.e. $x \leq p$. Hence $p$ is a minimal element of $A$. Conversely if $p \in A$ is a minimal element of $A$, then for any $x \in A$, $x \leq p$ is not valid, implying $x$ does not divide $p$. Since $A$ contains all positive integers other than 1, we conclude that no nonzero integers other than 1 can ever divide $p$. Hence $p$ is a prime number.
Practice Class 9 Solutions

Question 1.
(a) and (b)

(c) or the minimal representation:

(d)

(e)

(f)

Question 2.

<table>
<thead>
<tr>
<th>House</th>
<th>Garage</th>
<th>Light</th>
</tr>
</thead>
<tbody>
<tr>
<td>open</td>
<td>open</td>
<td>off</td>
</tr>
<tr>
<td>(a)</td>
<td>open</td>
<td>on</td>
</tr>
<tr>
<td>closed</td>
<td>open</td>
<td>on</td>
</tr>
<tr>
<td>closed</td>
<td>closed</td>
<td>off</td>
</tr>
</tbody>
</table>

or using the notation of Boolean algebra,
(b) Let the switch in the house be $x$ and the switch in the garage be $y$. Then the Boolean expression is

$$x' y + xy'$$

and the corresponding switching circuit is

![Switching Circuit Diagram]
Practice Class 10 Solutions

Question 1.

(i) (a) is 2nd order.  (b) is 1st order,  (c) is 3rd order,  (d) is 2nd order.

(ii) (a), (c) and (d) are homogeneous.

(iii) (a) $x^2 - 5x + 4 = 0$,  (c) $a^3 - 6a^2 + 11a - 6 = 0$,  (d) $a^2 - a - 1 = 0$.

(iv) (a) $x^2 - 5x + 4 = 0 \implies (x - 1)(x - 4) \implies x = 1, x = 4$.

Therefore the general solution is $S(n) = A(1^n) + B(4^n) = A + B(4^n)$.

Using the initial conditions, we have $S(0) = A + B = 2$ and $S(1) = A + 4B = 5$, and so $A = 1$ and $B = 1$.

So the proposed solution is $S(n) = 1 + 4^n$, for $n \geq 0$.

Check: $S(0) = 1 + 4^0 = 2$, and $S(1) = 1 + 4 = 5$, as required.

$$5S(n+1) - 4S(n)$$

$$= 5[1 + 4^{n+1}] - 4[1 + 4^n]$$

$$= 5 + 5(4^{n+1}) - 4 - 4(4^n)$$

$$= 1 + 5(4^{n+1}) - 4^{n+1}$$

$$= 1 + 4^{n+1}$$

$$= 1 + 4^{n+2} = S(n), \text{ as required.}$$

(c) $a^3 - 6a^2 + 11a - 6 = 0 \implies (a - 1)(a - 2)(a - 3) = 0 \implies a = 1, 2, 3$. Therefore the proposed solution is $R(n) = A(1^n) + B(2^n) + C(3^n)$ $= A + B(2^n) + C(3^n)$. Use initial conditions to find $A, B, C$.

$R(0) = A + B + C = 3$, $R(1) = A + 2B + 3C = 6$, and $R(2) = A + 4B + 9C = 14$.

Solving this system of simultaneous linear equations gives $A = B = C = 1$.

So the proposed solution is $R(n) = 1 + 2^n + 3^n$, for $n \geq 0$.

Check $R(0) = 1 + 2^0 + 3^0 = 3$, $R(1) = 1 + 2^1 + 3^1 = 6$ and $R(2) = 1 + 2^2 + 3^2 = 14$.

$$R(n+3) - 6R(n+2) + 11R(n+1) - 6R(n)$$

$$= 1 + 2^{n+3} + 3^{n+3} - 6 - 6 \times 2^{n+2} - 6 \times 3^{n+2} + 11 + 11 \times 2^{n+1}$$

$$+ 11 \times 3^{n+1} - 6 - 6 \times 2^n - 6.3^n$$

$$= [2^{n+3} - 6 \times 2^{n+2} + 11 \times 2^{n+1} - 6 \times 2^n] + [3^{n+3} - 6.3^{n+2} + 11 \times 3^{n+1} - 6 \times 3^n]$$

$$= 2^n[2^3 - 6 \times 2^2 + 11 \times 2 - 6] + 3^n[3^3 - 6 \times 3^2 + 11 \times 3 - 6]$$

$$= 2^n[0] + 3^n[0] = 0, \text{ as required.}$$
(d) \(x^2 - x - 1 = 0 \implies x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}\). So \(F_k = A \left(\frac{1 + \sqrt{5}}{2}\right)^k + B \left(\frac{1 - \sqrt{5}}{2}\right)^k\).

Use initial conditions: \(F_0 = 1 \implies A + B = 1\) and \(F_1 = 1 \implies A \left(\frac{1 + \sqrt{5}}{2}\right) + B \left(\frac{1 - \sqrt{5}}{2}\right) = 1 \implies \sqrt{5}A - \sqrt{5}B = 1\). Solving, gives \(A = \frac{5 + \sqrt{5}}{10}\) and \(B = \frac{5 - \sqrt{5}}{10}\).

Check Exercise.

Question 2.

(a) The characteristic equation is
\[x^2 + 6x + 9 = 0\]
i.e. \((x + 3)^2 = 0\), we have a repeated root \(x = -3\).

Hence the general solution is \(f(n) = (A + Bn)(-3)^n\), where \(A\) and \(B\) are constants.

Apply the initial conditions:
\[f(0) = A = 1\]
and \(f(1) = (A + B)(-3) = 0\), so \(A + B = 0\).

Therefore \(A = 1\) and \(B = -1\), and the solution is
\[f(n) = (1 - n)(-3)^n.\]

(b) The characteristic equation is
\[x^3 - 3x^2 + 4 = 0.\]

We guess \(x = -1\) as one root, the others can be obtained by guesswork or long division we have
\[x = -1\) and \(x = 2\) (repeated)
[i.e. \(x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2\)]

Hence the general solution is
\[A_n = (A + Bn)2^n + C(-1)^n\]

\(A, B, C\) all constant.
Practice Class 11 Solutions

Question 1.

(i)

\[
S(k + 2) - 6S(k + 1) + 9S(k) = 2
\]  

(1)

The associated homogeneous equation is

\[
S(k + 2) - 6S(k + 1) + 9S(k) = 0.
\]  

(2)

The characteristic equation is \(a^2 - 6a + 9 = 0 \Rightarrow (a - 3)^2 = 0\). So the root is 3 with multiplicity 2. Therefore the general solution to equation (2) is

\[
S(k) = (A + Bk)3^k, \text{ for } k \geq 0.
\]

We now seek a particular solution to (1): try \(S(k) = C\) for some \(C \in \mathbb{R}\). Substituting in (1) gives \(C - 6C + 9C = 2 \Rightarrow C = \frac{1}{2}\).

So the general solution to (1) is

\[
S(k) = (A + Bk)3^k + \frac{1}{2}, \text{ for } k \geq 0
\]

Check

\[
S(k + 2) - 6S(k + 1) + 9S(k)
\]

\[
= \left[(A + B(k + 2))3^{k+2} + \frac{1}{2}\right] - 6 \left[(A + B(k + 1))3^{k+1} + \frac{1}{2}\right] + 9 \left[(A + Bk)3^k + \frac{1}{2}\right]
\]

\[
= 3^k \left[k(9B - 18B + 9B) + (9A - 18A + 18B + 9A - 18B)\right] + \frac{1}{2} - 3 + \frac{9}{2}
\]

\[
= 2, \text{ as required.}
\]

Therefore \(S(k) = (A + Bk)3^k + \frac{1}{2}\) is a solution to (1).

(ii)

\[
T(n + 1) - 5T(n) = 7^n
\]  

(3)

The associated homogeneous equation is

\[
T(n + 1) - 5T(n) = 0
\]  

(4)

The characteristic equation is \(a - 5 = 0\), and so \(a = 5\) is the root. Therefore the general solution to (4) is

\[
T(n) = A5^n \text{ for } n \geq 0.
\]
We try $T(n) = C7^n$ as a particular solution to (3). Then $C7^{n+1} - 5C7^n = 7^n \implies C = \frac{1}{2}$.

Therefore the general solution to equation (3) is

$$T(n) = A5^n + \frac{7^n}{2}, \text{ for } n \geq 0.$$ 

The initial condition is $T(0) = \frac{9}{2}$. Substituting this gives

$$\frac{9}{2} = A + \frac{1}{2}$$

which gives $A = 4$. So the solution is

$$T(n) = 4 \times 5^n + \frac{7^n}{2}, \quad n \geq 0.$$ 

Check:

$$T(n + 1) - 5T(n)$$

$$= 4 \times 5^{n+1} + \frac{7^{n+1}}{2} - 5 \left[ 4 \times 5^n + \frac{7^n}{2} \right]$$

$$= 5^{n+1} (4 - 4) + 7^n \left( \frac{7}{2} - \frac{5}{2} \right)$$

$$= 7^n,$$ as required.

So $T(n) = 4 \times 5^n + \frac{7^n}{2}$ is a solution to (3).

(iii) $P(n + 2) - 4P(n + 1) + 4P(n) = n^2$ (5)

The associated homogeneous equation is

$$P(n + 1) - 4P(n + 1) + 4P(n) = 0$$ (6)

The characteristic equation is $x^2 - 4x + 4 = 0 \implies x = 2$ is the root with multiplicity 2.

So the general solution to (6) is

$$P(n) = (A + Bn)2^n, \quad n \geq 0.$$ 

We try $P(n) = Cn^2 + Dn + E$ as a particular solution to (5):

$$\implies C(n + 2)^2 + D(n + 2) + E - 4 \left[ C(n + 1)^2 + D(n + 1) + E \right] + 4 \left[ Cn^2 + Dn + E \right] = n^2$$

Comparing coefficients gives $C = 1, D = 4$ and $E = 8$. So the general solution of (5) is

$$P(n) = (A + Bn)2^n + n^2 + 4n + 8, \quad A, B \in \mathbb{R}, \quad n \geq 0.$$
**Question 2.**

We are given that “the distance it travels in each second is equal to twice the distance it travelled in the previous one”. The distance travelled in the \( r \)th second is

\[ a_r - a_{r-1} \]

and the distance travelled in the \((r - 1)\)th second is

\[ a_{r-1} - a_{r-2} \]

Hence

\[ a_r - a_{r-1} = 2(a_{r-1} - a_{r-2}) \]

i.e. \( a_r - 3a_{r-1} + 2a_{r-2} = 0 \)

This is a second order, linear, constant coefficient, homogeneous recurrence relation. The characteristic equation is \( x^2 - 3x + 2 = 0 \) with distinct roots \( x = 2, 1 \).

The general solution is

\[ a_r = A_1 + A_2 2^r \]

Apply the initial conditions

\[ a_0 = A_1 + A_2 = 3 \]

and \( a_1 = A_1 + 2A_2 = 10 \)

Solve for \( A_1 \) and \( A_2 \) to get \( A_1 = -4, A_2 = 7 \).

The solution is

\[ a_r = 7 \times 2^r - 4. \]

**Question 3.**

The deposit doubles each year with an initial deposit of 100, the deposits form a geometric sequence with ratio 2 and first term 100. So the amount deposited in the \( n \)th year is \( 100 \times 2^n \) dollars. Thus the balance after \( n \) years is

\[ B(n) = \frac{10}{100}B(n-1) + B(n-1) + 100 \times 2^n \]

i.e. \( B(n) - \frac{11}{10}B(n-1) = 100 \times 2^n \), with \( B(0) = 100 \). This is easily solved to give

\[ B(n) = \frac{2000}{9}2^n - \frac{1100}{9} \left( \frac{11}{10} \right)^n \]
Supplementary Practice Class - Revision Solutions

Question 1.

The following is the corresponding graph of the problem. Note that there are only two vertices of odd degree, $D$ and $E$. Hence by Euler’s theorem, there is an Eulerian path. Using Fleury’s algorithm, beginning at vertex $D$, the path is

$$k, p, o, m, g, h, n, i, f, a, d, e, b, c.$$

Question 2.

$$[(a + bc) - ((c/d) + e)] + efg.$$

Question 3.

The characteristic equation is

$$6x^2 - 5x + 1 = 0$$

The roots are $x = \frac{1}{2}$ and $\frac{1}{3}$ these are distinct so the general solution is

$$f(n) = A_1 \left(\frac{1}{2}\right)^n + A_2 \left(\frac{1}{3}\right)^n$$

Now apply the initial conditions

$$f(0) = A_1 + A_2 = 1$$

and $f(1) = \frac{1}{2}A_1 + \frac{1}{3}A_2 = 0$

Solve for $A_1$ and $A_2$ to get $A_1 = -2$, $A_2 = 3$. 
Therefore the particular solution is
\[ f(n) = 3 \left( \frac{1}{3} \right)^n - 2 \left( \frac{1}{2} \right)^n, \quad n \geq 2. \]

**Question 4.** An ordered edge list is
\[
\begin{align*}
  e_1 &= \{v_1, v_2\} & e_2 &= \{v_4, v_5\} & e_3 &= \{v_2, v_4\} \\
  e_4 &= \{v_5, v_6\} & e_5 &= \{v_6, v_7\} & e_6 &= \{v_1, v_5\} \\
  e_7 &= \{v_3, v_4\} & e_8 &= \{v_1, v_4\} & e_9 &= \{v_7, v_8\} \\
  e_{10} &= \{v_4, v_6\} & e_{11} &= \{v_2, v_3\} & e_{12} &= \{v_5, v_7\} \\
  e_{13} &= \{v_6, v_8\} & e_{14} &= \{v_3, v_8\}
\end{align*}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
N(1) & N(2) & N(3) & N(4) & N(5) & N(6) & N(7) & N(8) & E & \text{weight} \\
\hline
\text{initially} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \phi & 0 \\
\text{After Step 1} & 1 & 1 & 3 & 4 & 5 & 6 & 7 & 8 & \{e_1\} & 1 \\
\text{After Step 2} & 1 & 1 & 3 & 4 & 4 & 6 & 7 & 8 & \{e_1, e_2\} & 2 \\
\text{After Step 3} & 1 & 1 & 3 & 1 & 1 & 6 & 7 & 8 & \{e_1, \cdots, e_3\} & 4 \\
\text{After Step 4} & 1 & 1 & 3 & 1 & 1 & 1 & 7 & 8 & \{e_1, \cdots, e_4\} & 7 \\
\text{After Step 5} & 1 & 1 & 3 & 1 & 1 & 1 & 1 & 8 & \{e_1, \cdots, e_5\} & 10 \\
\text{After Step 7} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 8 & \{e_1, \cdots, e_5, e_7\} & 14 \\
\text{After Step 9} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \{e_1, \cdots, e_5, e_7, e_9\} & 19 \\
\hline
\end{array}
\]

The weight of the minimal spanning tree found here is 19. (You should draw the tree.)

**Question 5.**

The associated homogeneous recurrence relation is
\[ S(n+2) - 2S(n+1) + S(n) = 0 \]
which has characteristic equation \((a - 1)^2 = 0\). So 1 is a double root, and the general solution to the homogeneous recurrence relation is
\[ S(n) = (A + Bn)1^n = A + Bn, \quad n \geq 0. \]

As the right hand side is a constant, we would normally try as a particular solution \(S_p(n) = C\), but as 1 is a double root of the left hand side and a root of the right hand side, we try \(S_p(n) = Cn^2\). So
\[
\begin{align*}
  S(n+2) - 2S(n+1) + S(n) &= 4 \\
  \implies C(n+2)^2 - 2C(n+1)^2 + Cn^2 &= 4 \\
  \implies C(n^2 + 4n + 4) - 2C(n^2 + 2n + 1) + Cn^2 &= 4 \\
  \implies 4C - 2C &= 4 \\
  \implies C &= 2
\end{align*}
\]
So the general solution to the non–homogeneous recurrence relation is

\[ S(n) = A + Bn + 2n^2. \]

Using initial conditions we obtain the particular solution to the non–homogeneous recurrence relation:

\[ S(n) = 2 - 2n + 2n^2. \]

Check: Exercise.
Question 6.

\[ (x+y)(x+z)y'z \]

\[ x \rightarrow (x+y) \rightarrow (x+z)y'z \]

\[ y \rightarrow y' \]

\[ z \rightarrow c \rightarrow s \rightarrow yz+y'z' \]

Question 7.

\[ x \]

Question 8.

Note that

\[
|f(n)| = |2n^3 + 3n^2 \log n + 2n - 1| \\
= |2n^3 + 3n^2 \log n + (2n - 1)| \\
= 2n^3 + 3n^2 \log n + 2n - 1 \text{ because all the terms are positive since } n \geq 1 \\
\leq 2n^3 + 3n^2 + 2n \text{ as } \log n \leq n \text{ for } n \geq 1 \text{ and } 2n - 1 \leq 2n \\
\leq 7n^3, \text{ for } n \geq 1.
\]

Also

\[
|f(n)| = |2n^3 + 3n^2 \log n + 2n - 1| \\
= 2n^3 + 3n^2 \log n + 2n - 1 \text{ as } n \geq 1 \\
\geq 2n^3 + 3n^2 \log n \text{ as } n \geq 1 \text{ makes } 2n - 1 \geq 0 \\
\geq 2n^3, \text{ as } 3n^2 \log n \geq 0.
\]

Therefore choose \( c = 8, \ d = 2 \) and \( M = 1 \). Then \( f(n) = \Theta(n^3) \).
How to Submit Your Assignments

The assessment consists of

- Six assignments worth a total of 600 marks (30% from the final mark);
- Final examination worth 100 marks (70% from the final mark).

To pass this unit, you must accumulate at least 50% in each of the two components

- submit reasonable attempts on at least three assignments such that you’ll accumulate at least 300 marks (15% from the final mark);
- achieve at least 50 marks (35% from the final mark) in the examination.

There are six assignments for this unit, worth 100 (5%) marks each. You are expected to hand in solutions to each assignment to get a better mark. Sample solutions will be posted on the unit’s web site and/or will be send as feedback to students. It is important that you hand in the assignments at the nominated times, or at least make contact with us to let us know if there is some problem.

The standard procedure for submission is by bringing your work to the lecturer in the class or by dropping it in the dedicated box on the second floor in Booth building (mostly for internals). The external students must send to us a pdf scan of their work. This can be done

1. Electronically, through our server turing by clicking our secure connection, then

- select the appropriate assignment number and upload your file(s).


2. Alternatively, one may print the cover sheet from the UNE e-submission web-site, sign it, attach it to the assignment and send it by post.

<table>
<thead>
<tr>
<th>ASSIGNMENT</th>
<th>POSTING DATE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8th of July</td>
</tr>
<tr>
<td>2</td>
<td>22nd of July</td>
</tr>
<tr>
<td>3</td>
<td>5th of August</td>
</tr>
<tr>
<td>4</td>
<td>27th of August</td>
</tr>
<tr>
<td>5</td>
<td>10th of September</td>
</tr>
<tr>
<td>6</td>
<td>24th of September</td>
</tr>
</tbody>
</table>

Please read and obey our University’s plagiarism policy. While the University’s policy allows us 30 days for marking an assignment, we’ll try to do our best to mark it and send back to you the feedback as soon as possible. Notice that we can not send back the marking (and solution) before everybody submitted.
ASSIGNMENT 1

Question 1
Let \( A = \{2n + 1 \mid n \in \mathbb{Z}, \ 0 \leq n \leq 4\} \) and \( B = \{2, 3, 4\} \).

(a) Find \( A \cup B, \ A \cap B \) and \( A \times B \).

(b) Find the power set of \( B, \ \mathcal{P}(B) \).

(c) Find a partition of \( B \) that has 3 elements.

Question 2
Use the set property \( A - B = A \cap B' \) to show that for all sets \( A, B \) and \( C \)

\[
(A - B) - C = (A - C) - B.
\]

Question 3
Use mathematical induction to prove that \( 2^{3n} - 1 \) is divisible by 7 for each integer \( n \geq 1 \).

Question 4
Let \( f(x) = 3x^3 + 2x^2 - 5x - 4 \). Evaluate \( f(1) \) and \( f(-2) \) by Horner’s method.

Question 5
Let \( f : \mathbb{R} \to \mathbb{R} \) be given by \( f(x) = x^3 + 2x^2 - x + 2 \). Prove, from first principles, that \( f(x) = O(x^3) \).

Question 6
Let \( f : \mathbb{N} \to \mathbb{R} \) be defined by

\[
f(n) = \frac{n^2 + 2 \log_2 n}{n + 1}.
\]

Prove, from first principles, that \( f(n) = \Theta(n) \).
ASSIGNMENT 2

Question 1

Use Binary Search to seek the letter H in the list

\{ A, B, D, E, G, H, I, J, K \}.

Give the reduced list at each step and mark the corresponding middlemost elements with boxes. How many comparisons are performed in total?

Question 2

(a) Prove \( p \to q \lor r \) is equivalent to \( p \land (\sim r) \to q \).

(b) Prove that \( p \to p \lor q \) is a tautology.

Question 3

Use a truth table to determine whether the argument form

\[ p \land q \to \sim r, \ p \lor (\sim q), \ \sim q \to p, \ \therefore \sim r \]

is valid or not.

Question 4

Show the following argument form

\[ \sim r \to p, \ (\sim p) \lor q, \ \sim s \to (\sim p) \land (\sim r), \ (\sim p) \land r \to (\sim s) \lor t, \ \sim q, \ \therefore t \]

is valid by deducing the conclusion from the premises step by step through the use of basic inference rules or laws of logic.

Question 5

(a) Express the following statement as a sentence in English. Is this statement true? Explain your answer.

\[ \forall n \in \mathbb{N}, \ \exists m \in \mathbb{N}, \ (n = m^2 + 1) \]

(b) For the following statement, find its

(i) negation

(ii) contrapositive.

\[ \forall x \in \mathbb{R}, \ (x^2 > 1) \to (x < -1) \]
ASSIGNMENT 3

Question 1

(a) Sort with the Insertion Sort algorithm the following list:

\[ E, C, A, D, B, F \]

How many comparisons will be performed? Give the intermediate list whenever a number has been completely inserted.

(b) Sort the list in (i) with the Quicksort algorithm, using the first element as a pivot. Underline the pivot elements, and use an asterisk (*) to mark the elements in their final positions. How many comparisons are needed in this case?

Question 2

Consider the following graph

Find the following if (they exist)

(a) An Eulerian path.

(b) A circuit of exactly 5 edges that goes through vertex \( a \) and edge \( \{e, f\} \).

(c) A Hamiltonian path that goes through the edges \( \{c, d\} \) and \( \{e, f\} \).

(d) A Hamiltonian circuit that goes through the edges \( \{c, d\} \) and \( \{e, f\} \).

(e) An Eulerian circuit.

Is the graph Eulerian?
Question 3

(a) Group the following graphs into isomorphism classes, that is, put graphs that are isomorphic to each other into the same class and keep graphs that are not isomorphic to each other in different classes.

(b) For each isomorphism class give an (isomorphic) invariant which distinguishes that class from the others.

(c) Let the vertices of $G_1$ and $G_5$ be denoted by

Are graphs $G_1$ and $G_5$ isomorphic to each other? If yes, then provide the isomorphism mapping that associates the corresponding vertices of the two graphs.
Question 4
For the following graph

(a) Give the adjacency matrix for the graph.
(b) How many walks from vertex $v_2$ to vertex $v_3$ are there which are of length 3?

Question 5
In each case, could a graph with the following properties be a tree? If your answer is yes, draw a tree with these properties. If your answer is no, prove that it couldn’t be a tree.

(a) 8 vertices, with 2 vertices of degree 1, 1 vertex of degree 2 and 5 vertices of degree 3.
(b) 7 vertices, with 2 vertices of degree 1, 3 vertices of degree 2 and 2 vertices of degree 3.
(c) 6 vertices, with 4 vertices of degree 1 and 2 vertices of degree 3.
ASSIGNMENT 4

Question 1

(a) Draw a binary tree to describe the following expressions.

(i) \( a/(a + b) \times (b - c)) \).

(ii) \(((a/b)/c) - (b \times (a/c)) \times ((b \times (c + d)) + (b/c))) \).

(b) Write down the vertex sequence for the postorder traversal, i.e. in the postfix notation, for the binary tree found in (a) part (ii).

Question 2

Sort, using a binary tree, the following list of numbers

7, 12, 2, 9, 4, 1, 19, 16, 8, 10.

Question 3

Eight towns, which we label A to H, are to be connected by a telephone network. The network cable costs $15,000 per kilometer to lay. The required connections and distances between the towns for which we require a connection are given in the table below. Use Kruskal’s algorithm to find the network of minimal cost. What is the minimum cost?

<table>
<thead>
<tr>
<th>Connection</th>
<th>Distance (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A to B</td>
<td>5</td>
</tr>
<tr>
<td>A to D</td>
<td>6</td>
</tr>
<tr>
<td>B to C</td>
<td>8</td>
</tr>
<tr>
<td>B to D</td>
<td>11</td>
</tr>
<tr>
<td>B to E</td>
<td>9</td>
</tr>
<tr>
<td>C to E</td>
<td>5</td>
</tr>
<tr>
<td>C to H</td>
<td>5</td>
</tr>
<tr>
<td>D to E</td>
<td>7</td>
</tr>
<tr>
<td>E to G</td>
<td>16</td>
</tr>
<tr>
<td>E to F</td>
<td>13</td>
</tr>
<tr>
<td>F to G</td>
<td>9</td>
</tr>
<tr>
<td>G to A</td>
<td>11</td>
</tr>
<tr>
<td>G to H</td>
<td>7</td>
</tr>
</tbody>
</table>

Question 4
(a) Convert the following numbers to base 4, and then to the hexadecimal representation.

(i) 22

(ii) 321₃

(iii) 1110111.011₂

(b) Perform the following in base 7 (the numbers are in base 7)

(i) \[
\begin{array}{c}
3526 \\
+ \\
461 \\
\end{array}
\]

(ii) \[
\begin{array}{c}
3526 \\
- \\
461 \\
\end{array}
\]

**Question 5**

Define two binary relations from \(\mathbb{R}\) to \(\mathbb{R}\) as follows

\[
R = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x^2 + y^2 = 9\}
\]

\[
S = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x \leq y\}
\]

Graph \(R\) and \(S\) separately in the Cartesian plane.

**Question 6**

Let \(V\) denote the set of the vertices of the following graph \(G\)

we define a relation \(R\) on \(V\) by

\[(v, w) \in R \quad \text{iff} \quad \text{there exists at least one edge in } G \text{ which connects vertex } v \text{ to vertex } w \text{ directly}\]

(a) Is \(R\) reflexive? symmetric? transitive? Why?

(b) Is \(R\) an equivalence relation?
(c) Let graph $G'$ be given by

and let relation $R'$ on $V$ be defined by

$$(v, w) \in R' \iff \text{there exists at least one edge in } G' \text{ which connects vertex } v \text{ to vertex } w \text{ directly}$$

Is $R'$ now an equivalence relation? What is the induced partition of the set $V$?
ASSIGNMENT 5

Question 1
Consider the “divides” relation \( R \) (or “\( \leq \)”) on the following set

\[ A = \{2, 3, 4, 5, 12, 15, 18\} \]

i.e. for any \( x, y \in A \): \( xRy \) (or \( x \leq y \)) if and only if \( x \mid y \), viz. \( y \) is divisible by \( x \).

(a) Draw a directed graph to represent the relation \( R \).

(b) Explain from the digraph in (i) that \( R \) is a partial order relation.

(c) Draw the Hasse diagram for \( R \).

(d) Find all the maximal and minimal elements of \( A \) under the partial ordering of \( R \).

(e) Does \( A \) have a greatest element and a least element under the partial order relation \( R \)? Find them if any, and explain why if not.

Question 2
Let \( S = \{0, 1\} \). We define on \( S \) two binary operations, “+” and “\( \cdot \)”, and one unary operation “\( ' \)” as follows.

\[
\begin{align*}
0 + 0 &= 0, & 0 + 1 &= 1, & 1 + 0 &= 1, & 1 + 1 &= 1; \\
0 \cdot 0 &= 0, & 0 \cdot 1 &= 0, & 1 \cdot 0 &= 0, & 1 \cdot 1 &= 1; & 0' &= 1, & 1' &= 0.
\end{align*}
\]

Verify B5 of the definition of a Boolean algebra for \( (S, +, \cdot, ', 0, 1) \). Also verify property P4 for \( (S, +, \cdot, ', 0, 1) \).

That is, verify

(a) \( a + a' = 1 \)

(b) \( a \cdot a' = 0 \)

(c) \( a + a \cdot b = a \)

for all \( a, b \in S \) by completing the following tables

<table>
<thead>
<tr>
<th>( a )</th>
<th>( a' )</th>
<th>( a + a' )</th>
<th>( a \cdot a' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>( 1 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>( 1 )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td>a · b</td>
<td>a + a · b</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>-------</td>
<td>----------</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
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<tr>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Please explain briefly how you draw the conclusion from the tables.

**Question 3**

The table below specifies a Boolean function \( f: S \times S \times S \rightarrow S \)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>( f(x, y, z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
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<td>1</td>
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<tr>
<td>0</td>
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<td>1</td>
<td>0</td>
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<td>1</td>
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<td>1</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

where \( S = \{0, 1\} \). Construct a Boolean expression corresponding to this function.

**Question 4**

Design an electronic voting machine that allows a four person committee to vote (each person has one vote) so that a light will go on if a majority of 3 or more vote yes. Simplify the Boolean expression first, if possible. Then draw a switching system for the Boolean expression.

**Question 5**

Give the gate implementation of the following Boolean expressions in their *current* form.

(a) \((x + y)(y + x')\)

(b) \(x + y + (xy)'\)

**Question 6**

Use Karnaugh maps to find a minimal representation for the following Boolean expressions

(a) \(x'y'z + x'y'z' + x'y'z\)

(b) \(w'xyz + w'xy'z + wxy'z + w'xy'z' + wxy'z + w'xy'z + w'x'y'z\)
ASSIGNMENT 6

Question 1
A mapping \( f: \mathbb{N} \rightarrow \mathbb{N} \) is defined by
\[
\begin{align*}
f(0) &= 1, \\
f(n) &= 1 + \sum_{j=0}^{n-1} jf(j), \quad n \geq 1.
\end{align*}
\]
Find \( f(4) \).

Question 2
(a) Is \( a_{k+1} + a_k^2 + k^2 = 0 \) a linear constant coefficient recurrence relation? Why? Find \( a_2 \) and \( a_4 \) if \( a_0 = 1 \).

(b) Find the general term \( f(n), n \geq 1 \), for the recurrence relation
\[
f(n) = 3f(n-1), \quad \text{with} \quad f(1) = 4.
\]

Question 3
Find the general solution for the following recurrence relation
\[
a_{n+2} + a_{n+1} - 12a_n = 0.
\]

Question 4
Find the particular solution of the following difference equation
\[
a_{n+2} - 2a_{n+1} - 63a_n = 64n, \quad n \geq 0
\]
with \( a_0 = 0 \) and \( a_1 = 7 \).

Question 5
Find the general solution of \( b_{n+2} - 4b_{n+1} + 4b_n = 8 \times 2^n \) for \( n \in \mathbb{N} \).